A Three-Factor Defaultable Term Structure Model

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Although credit risk has always been an important concern, only relatively recently have banks and other financial institutions become more aware of the weaknesses of their traditional credit risk exposure calculation techniques. New instruments like credit derivatives also require new models for credit risk pricing. Thus, it is not surprising that there have been many theoretical developments in credit risk research in the past several years.

There are two basic approaches for modeling default risks in bonds, structural models and reduced-form models. The structural approach dates to Merton [1974]. It assumes that the dynamics for the value of the assets of a firm across time can be described by a diffusion stochastic process and that the defaultable security can be regarded as a contingent claim on the value of the assets of the firm. Apart from the standard assumptions of continuous-time no-arbitrage models, there are many shortcomings of this model. The liabilities of the firm are supposed to consist of only a single class of debt; the debt has a zero coupon; bankruptcy is triggered only at the maturity of the debt; bankruptcy is costless; and interest rates are assumed to be constant over time.[1] Thus, the assumptions of the Merton model are highly stylized versions of reality.

As Jones, Mason, and Rosenfeld [1984] and Kim, Ramaswamy, and Sundaresan [1992, 1993] point out, Merton's model does not generate the levels of yield spreads that are observed in the market. Rather, the model is unable to generate yield spreads in excess of 120 basis points, while over the 1926-1986 period the yield spreads of AAA-rated corporates ranged from 15 to 215 basis points. This inability to account for the magnitude of yield spreads is the motivation for a great number of articles during the last two decades that generalize Merton's risky debt pricing model.

Black and Cox [1976] incorporate classes of senior and junior debt; Geske [1977] considers coupon bonds; and Ho and Singer [1984] allow for different maturities of debt. Because the empirical literature shows that credit spread curves can be flat or even downward sloping, that short-term debt often does not have zero credit spreads, and that sudden drops in the value of firms are possible, Zhou [1997] incorporates a jump process for the underlying asset value. Shimko, Tejima, and Van Deventer [1993] allow for stochastic nondefaultable short rates as in Vasicek [1977] with interest rates that follow a mean-reverting process with constant volatility. Kim, Ramaswamy, and Sundaresan [1992, 1993] use a Cox-Ingersoll-Ross-like model for the stochastic non-defaultable short rates. In addition, they study the role of call features in corporate and Treasury bonds and find that the call feature is relatively more valuable in Treasuries than in corporate issues.

Longstaff and Schwartz [1995] develop a model similar to Shimko, Tejima, and Van

Deventer [1993] except for the bankruptcy procedure. While most researchers using option pricing frameworks assume bankruptcy is triggered at the moment the value of the firm reaches the value of the debt, Longstaff and Schwartz model default as the time when the value of the debt reaches some constant threshold value K that serves as a distress boundary, i.e., the default time τ can then be expressed formally as $\tau = \inf\{t \geq 0 : V(t) \leq K\}$, the first time at which $V(t)$ (the value of the firm's assets at time t) crosses the lower bound K. If the value of the assets drops to this level, default is triggered; some form of restructuring occurs; and the remaining assets of the firm are allocated among the firm's claimants.

Implicit in this formulation is the assumption that once this level is reached, default occurs on all outstanding liabilities at the same time. Thus, contrary to Merton's model, default can occur prior to maturity. Because of the complexities of these extensions, often a closed-form solution cannot be obtained and numerical procedures must be used.

Despite the immense effort exerted to generalize Merton's methodology, all these structural models have only limited success explaining the behavior of prices of debt instruments and credit spreads. This has led to attempts to use models that make more direct assumptions about the default process. These alternative approaches, called reduced-form models, don't consider the relation between default and asset value in an explicit way, but rather model default as a stopping time of some given hazard rate process, i.e., the default process is specified exogenously.

This achieves two effects. The first is that the model can be applied to situations where the underlying asset value is not observable. Second, the default time is unpredictable, so the behavior of credit spreads for short maturities can be captured more realistically. In addition, the approach is very tractable and flexible so as to fit the observed credit spreads.

Some important examples of this approach are Jarrow and Turnbull [1995], who compare the bankruptcy process to a spot exchange rate process; Lando [1994, 1996, 1998] who uses Cox processes, which can be thought of as Poisson processes with a random intensity, to model prices of credit-risky debt; and Duffie and Singleton [1997, 1998], who show that assets can be valued under the risk-adjusted probability measure by discounting the non-defaultable payoff on the debt by a discount rate that is adjusted for the parameters of the default process. Schönbucher [1996] presents a generalization in a Heath-Jarrow-Morton framework that allows for restructuring of defaulted debt, multiple defaults, and loss quotas that are not predictable.

While the reduced-form models have attractive properties, their main drawback is lack of a link between firm value and corporate default. This article develops a three-factor defaultable term structure model for the pricing of a wide range of risky debt contracts and derivatives. It combines structural and reduced-form models.

One of the factors that determine the credit spread is the so-called uncertainty process, which can be understood as an aggregation of all information on the quality of the firm currently available: The greater the value of the uncertainty process, the poorer the quality of the firm. A similar idea to this uncertainty process was first introduced by Cathcart and El-Jahel [1998] in the form of a "signaling process" that explicitly drives default.

Our model differs from Cathcart and El-Jahel in several ways. First, we assume the underlying short rate follows either a mean-reverting Hull-White process or a mean-reverting square root process with time-dependent mean reversion level. The uncertainty or signaling process is assumed to follow a mean-reverting square root process. We start our considerations in the "real world" instead of the "risk-adjusted world," and use Girsanov's theorem for the change of measure. Finally, besides modeling the non-defaultable short rate and the uncertainty index, we directly model the short rate spread process. We assume that the spread between a defaultable and a non-defaultable bond is driven to a considerable extent by the uncertainty index, but that there may be additional factors that influence the level of the spreads: at minimum the contractual provisions, liquidity, and the premium demanded in the market for similar instruments. We can easily relate credit spreads to business cycles by replacing the uncertainty index by some index of macroeconomic variables without any change of the theoretical framework.

Our approach seems to be reasonable in that credit spreads provide useful observable information on data upon which pricing models can be based. In addition, the model can be fitted directly to match the actual process followed by interest rate credit spreads. The analytical solution obtained for defaultable bonds can be implemented easily in practice, as all the variables and parameters can be deduced from market data.

We derive a closed-form solution for pricing pure discount defaultable bonds and analyze it. Some examples demonstrate the behavior of the credit spread term structure. In addition, we show how the model can be applied to price options on defaultable bonds and credit derivatives by using tree-based methods and how to calibrate the model to market data.

I. THE MODEL

We want to develop a relative pricing theory based on the no-arbitrage assumption to determine fair values for defaultable securities and credit derivatives. As usual, we assume that markets are frictionless and perfectly competitive, that trading takes place continuously, that there are no taxes, transaction costs, or informational asymmetries, and that investors act as price takers. We fix a terminal time horizon T^*. Uncertainty in the financial market is modeled by a complete probability space $(\Omega, F, \mathbf{P})$ and a filtration $\mathbf{F} = (F_t)_{0 \leq t \leq T^*}$ of the three-dimensional standard Brownian motion $W(t)' = [W_r(t), W_s(t), W_u(t)]$ ($'$ denotes the transpose of a vector v) satisfying the usual conditions (i.e., F_0 is trivial and includes all the $\mathbf{P}$–null sets of F, and the filtration $\mathbf{F}$ is right-continuous). In addition, we assume $F_{T^*} = F$. All processes are defined in the probability space $(\Omega, F, \mathbf{P})$.

Furthermore, we make several assumptions:

Assumption 1. *The dynamics of the non-defaultable short rate are given by the stochastic differential equation:*

$$dr(t) = [\theta_r(t) - a_r r(t)]\, dt + \sigma_r r(t)^\beta dW_r(t), \quad t \geq 0, \qquad (1)$$

where a_r and σ_r are positive constants, $\beta = 0$ or $1/2$, and θ_r is a non-negative valued deterministic function.

This specification implies that the current rate $r(t)$ is pulled toward $\theta_r(t)/a_r$ with a speed of adjustment a_r, and if $\beta = 1/2$ the instantaneous variance of the change in the rate is proportional to its level.

Assumption 2. *The development of the uncertainty index is given by the stochastic differential equation:*

$$du(t) = [\theta_u - a_u u(t)]\, dt + \sigma_u \sqrt{u(t)} dW_u(t), \quad t \geq 0, \qquad (2)$$

where a_u and σ_u are positive constants, and θ_u is a non-negative constant.

Assumption 3. *The dynamics of the short rate spread (the short rate spread is supposed to be the defaultable short rate minus the non-defaultable short rate) is given by the stochastic differential equation:*

$$ds(t) = [b_s u(t) - a_s s(t)]\, dt + \sigma_s \sqrt{s(t)} dW_s(t), \quad t \geq 0, \qquad (3)$$

where a_s, b_s and σ_s are positive constants.

Additionally, we assume that the Brownian motions W_r, W_s, and W_u are correlated according to $Cov[dW_r(t), dW_s(t)] = \rho_{rs} dt$, $Cov[dW_r(t), dW_u(t)] = \rho_{ru} dt$, and $Cov[dW_s(t), dW_u(t)] = \rho_{su} dt$.

The assumption of the dynamics of the non-defaultable short rate can be replaced by any other specification. The uncertainty index can be interpreted as a measure for the quality of the obligor; the greater the value of the uncertainty index, the worse the quality. In the case of corporate bonds, one could use data from rating agencies or firm values to fit the uncertainty index and estimate the parameters. In the case of sovereigns, one could use data from rating agencies or some kind of macroeconomic data. The model admits various interpretations and is very flexible in capturing the characteristics of many different types of obligors.

The assumption of mean reversion of the uncertainty index is motivated by several empirical observations. Elsas et al. [1999], for example, mention that a downrating usually makes the management of a firm start taking actions to improve the firm's quality to reach the previous level again. In addition, the assumption of mean reversion of the short rate spread agrees with the prior beliefs of many academicians and practitioners. For example, Taurén [1999] points out that his empirical studies support the opinion that models that do not revert to the mean should be rejected. Our model generalizes previous models in that the mean reversion level is time-dependent and may depend on the uncertainty index. Thus, we take into consideration that the rating has a great impact on credit spreads.

We assume that there is an equivalent martingale measure $\mathbf{Q}$ under which the dynamics of $r(t)$, $s(t)$, and $u(t)$ are given by

$$dr(t) = [\theta_r(t) - \hat{a}_r r(t)]\, dt + \sigma_r r(t)^\beta d\hat{W}_r(t) \qquad (4)$$

$$ds(t) = [b_s u(t) - \hat{a}_s s(t)]\, dt + \sigma_s \sqrt{s(t)} d\hat{W}_s(t) \qquad (5)$$

$$du(t) = [\theta_u - \hat{a}_u u(t)]\, dt + \sigma_u \sqrt{u(t)} d\hat{W}_u(t), \quad 0 \leq t \leq T^* \qquad (6)$$

where $\hat{a}_i = a_i + \lambda_i \sigma_i^2$; and $\hat{W}_i(t) = W_i(t) + \int_0^t \gamma_i(\tau) d\tau$, $i = r, s, u$, $0 \leqslant t \leqslant T^*$ is a three-dimensional standard Brownian motion for the filtered probability space (Ω, F, $\mathbf{F}$, $\mathbf{Q}$) restricted to the time interval $[0, T^*]$.[2]

A contingent claim or derivative asset is a financial instrument whose payoffs are exactly determined by the payoffs on one or more underlying assets (for a more detailed definition, see Ingersoll [1987, pp. 50-51]). Following Duffie and Singleton [1997], a simple contingent claim is a pair (Z, τ) consisting of a non-negative F_τ – measurable random variable Z and a bounded stopping time $\tau \leqslant T^*$ at which Z is paid. Extending the Duffie-Singleton definition to cash flows at different times, a complex contingent claim is a triple (C, Z, τ) where C is a non-negative adapted payoff rate process $C = (C_t$ or $C(t)$: $0 \leqslant t < \tau)$; Z is a non-negative F_τ – measurable random variable; and $\tau \leqslant T^*$ is a bounded stopping time at which Z is paid. Logically the payoff rate is only paid prior to time τ.

There is accordingly a quite natural way of defining defaultable claims:

Definition. *A defaultable claim is a pair $[(C, G, T), (G', T')]$ of a complex contingent claim and a simple contingent claim. The underlying complex contingent claim (C, G, T) is the obligation of the issuer. The secondary simple contingent claim (G', T') defines the stopping time T' at which the issuer defaults and the payment G' received at default.*

This means that the actual claim (C, Z, τ) generated by a defaultable claim $[(C, G, T), (G', T')]$ is defined by $\tau = \min(T, T')$ and $Z = G 1_{T < T'} + G' 1_{T \geq T'}$. For simplification, we call (C, Z, τ) the defaultable claim. Without any loss of generality, we can assume that T is non-stochastic for all the pricing issues that we consider (i.e., T is some fixed maturity). The arbitrage-free price at time t, $F(t)$, of the defaultable claim (C, Z, τ) can be obtained as the discounted expected value of the future cash flows. This expectation has to be taken with respect to $\mathbf{Q}$. That is:

$$F(t) = E^{\mathbf{Q}} \left[\int_{[t,\tau)} \left(e^{-\int_t^u r(l) dl} C_u \right) du + e^{-\int_t^\tau r(l) dl} Z \mid \mathcal{F}_t \right], \quad 0 \leq t \leq \tau. \tag{7}$$

On the other hand, assuming the dynamics specified by Equations (1), (2), and (3), and recovery of market value this price must equal the expected value of riskless cash flows discounted at risky discount rates (see Das and Tufano [1996] and Duffie, Schroder, and Skiadas [1996]). That is:

$$F(t) = E^{\mathbf{Q}} \left[\int_{[t,T)} \left(e^{-\int_t^u (r(l)+s(l)) dl} C_u \right) du + e^{-\int_t^T (r(l)+s(l)) dl} G \mid \mathcal{F}_t \right], \quad 0 \leq t \leq \tau. \tag{8}$$

Under mild regularity conditions (see Duffie [1992]), and by using the Feynman-Kac formula, we get the lemma:

Lemma. *Assuming the dynamics specified by Equations (1), (2), and (3), the value at time t, $F(r, s, u, t)$, of a defaultable claim with underlying claim (C, G, T) is the solution to the partial differential equation:*

$$\begin{aligned} 0 = {} & \frac{1}{2} \left[\sigma_r^2 r^{2\beta} F_{rr} + \sigma_s^2 s F_{ss} + \sigma_u^2 u F_{uu} \right] \\ & + \rho_{rs} \sigma_r \sigma_s r^{\beta} \sqrt{s} F_{rs} + \rho_{ru} \sigma_r \sigma_u r^{\beta} \sqrt{u} F_{ru} + \rho_{us} \sigma_u \sigma_s \sqrt{su} F_{us} \\ & + [\theta_r(t) - \hat{a}_r r] F_r + [b_s u - \hat{a}_s s] F_s \\ & + [\theta_u - \hat{a}_u u] F_u + F_t - (r + s) F + C, \quad (r, s, u, t) \in \mathcal{R}^3 \times [0, T) \end{aligned} \tag{9}$$

with boundary condition:

$$F(r, s, u, T) = G, \quad (r, s, u) \in \mathcal{R}^3. \tag{10}$$

Similarly, assuming the dynamics specified by Equation (1), the time t value, $F(r, t)$, of a complex non-defaultable claim (C, G, T) is the solution to the partial differential equation:

$$0 = \frac{1}{2} \sigma_r^2 r^{2\beta} F_{rr} + [\theta_r(t) - \hat{a}_r r] F_r + F_t - rF + C, \quad (r, t) \in \mathcal{R} \times [0, T) \tag{11}$$

with boundary condition

$$F(r, T) = G, \quad r \in \mathcal{R}. \tag{12}$$

Hence, the time t value of a non-defaultable discount bond with maturity T, $P^*(t, T) = P^*(r, t, T)$, is given by (compare to Vasicek [1977], Hull and White [1990], and Hull [1997]):

$$P^*(r, t, T) = A^*(t, T) e^{-B^*(t,T) r}, \tag{13}$$

where $A^*(t, T)$ and $B^*(t, T)$ are defined by

$$B^*(t,T) = \begin{cases} \frac{1}{\hat{a}_r}\left[1-e^{-\hat{a}_r(T-t)}\right], & \text{if } \beta = 0, \\ \frac{1-e^{-\delta_r(T-t)}}{\kappa_1^{(r)}-\kappa_2^{(r)}e^{-\delta_r(T-t)}}, & \text{if } \beta = \frac{1}{2}, \end{cases} \text{, and,} \tag{14}$$

$$\ln A^*(t,T) = \begin{cases} \int_t^T \left(\frac{1}{2}\sigma_r^2 B^*(\tau,T)^2 - \theta_r(\tau) B^*(\tau,T)\right) d\tau & \\ = \ln\frac{P^*(0,T)}{P^*(0,t)} - B^*(t,T)\frac{\partial \ln P^*(0,t)}{\partial t} & \text{if } \beta = 0, \\ -\frac{\sigma_r^2}{4\hat{a}_r^3}\left(e^{-\hat{a}_r T} - e^{-\hat{a}_r t}\right)^2\left(e^{2\hat{a}_r t}-1\right), & \\ -\int_t^T \theta_r(\tau) B^*(\tau,T)\, d\tau, & \text{if } \beta = \frac{1}{2} \end{cases} \tag{15}$$

with $\delta_x = \sqrt{\hat{a}_x^2 + 2\sigma_x^2}$ and $\kappa_{1/2}^{(x)} = \frac{\hat{a}_x}{2} \pm \frac{1}{2}\delta_x$.

Equations (13), (14), and (15) define the price of a discount bond at a future time t in terms of the short rate at time t and the prices of non-defaultable discount bond prices today. The latter can be calculated from today's term structure. The partial differential $\frac{\partial \ln P^*(0,t)}{\partial t}$ in Equation (15) can be approximated by $\frac{\ln P^*(0,t+\varepsilon) - \ln P^*(0,t-\varepsilon)}{2\varepsilon}$ where ε is a small length of time.

In addition, if $\beta = 0$, the deterministic function $\theta_r(t)$ is given by

$$\theta_r(t) = f_t^*(0,t) + \hat{a}_r f^*(0,t) + \frac{\sigma_r^2}{2\hat{a}_r}\left(1 - e^{-2\hat{a}_r t}\right), \tag{16}$$

where $f^*(0, t)$ denotes the instantaneous forward rate as seen at time 0 for a contract maturing at time t. If $\beta = 1/2$, $\theta_r(t)$ can be obtained iteratively from

$$-\int_0^t \theta_r(\tau) B^*(\tau,t)\, d\tau = \ln A^*(0,t) \tag{17}$$

by time discretization.

II. THE PRICING OF DEFAULT-RELATED INSTRUMENTS

It is usually very hard to solve Equation (9) directly to obtain solutions for prices of bonds and other interest rate derivatives. Hence, we assume $\rho_{rs} = \rho_{ru} = \rho_{su} = 0$. Note that although W_s and W_u are independent, the short rate spread $s(t)$ and the uncertainty index $u(t)$ are correlated through the stochastic differential equation for the short rate spread.

Defaultable Discount Bonds

Applying the lemma to a defaultable discount bond yields the theorem:

Theorem. *Assuming the dynamics specified by Equations (1), (2), and (3), the value at time time t, $P(t, T) = P(r, s, u, t, T)$, of a defaultable discount bound is given by*

$$P(t,T) = A(t,T)\, e^{-B^*(t,T)r(t) - C(t,T)s(t) - D(t,T)u(t)}, \tag{18}$$

where

$$C(t,T) = \frac{1-e^{-\delta_s(T-t)}}{\kappa_1^{(s)} - \kappa_2^{(s)} e^{-\delta_s(T-t)}}, \tag{19}$$

$$D(t,T) = \frac{-2v'(t,T)}{\sigma_u^2 v(t,T)}, \tag{20}$$

$$\ln A(t,T) = \ln A^*(t,T) + \frac{2\theta_u}{\sigma_u^2}\ln\left|\frac{v(T,T)}{v(t,T)}\right|, \tag{21}$$

where $v(t, T)$ is defined in the appendix.

Proof. See the appendix.

Remark. *If we extend the definition of the uncertainty index from a constant parameter θ_u to a time-dependent parameter $\theta_u(t)$ the values of $\theta_u(t)$ can be determined iteratively from*

$$\ln\frac{A(0,t)}{A^*(0,t)} = -\int_0^t \theta_u(\tau) D(\tau,t)\, d\tau. \tag{22}$$

Although the bond pricing and credit spread formulas are rather complex and there are many parameters involved, it is easy to prove that they show realistic features: $\lim_{T\to\infty} P(r, s, u, t, T) = 0$; for $T > t$: $\lim_{r\to\infty} P(r, s, u, t, T) = \lim_{s\to\infty} P(r, s, u, t, T) = 0$ since $B^*(t, T) > 0$ and $C(t, T) > 0$, the price of a defaultable bond is a decreasing function of r and a decreasing function of s; furthermore, it is a convex function of r and a convex function of s.

The credit spread is defined as the difference between the yields of a defaultable and a corresponding non-defaultable bond. Since $A(t,T) = A^*(t,T)\, e^{-\int_t^T \theta_u D(\tau,T)d\tau}$, the defaultable bond pricing formula can be written as

$$P(r,s,u,t,T) = P^*(r,t,T)\, e^{-\int_t^T \theta_u D(\tau,T)d\tau - C(t,T)s - D(t,T)u} \tag{23}$$

Hence, the default premium of the defaultable bond is given by

$$P^*(r,t,T)\left(1 - e^{-\int_t^T \theta_u D(\tau,T)d\tau - C(t,T)s - D(t,T)u}\right). \tag{24}$$

If we denote the time t yield to maturity of a non-defaultable discount bond with maturity T by $R^*(r, t, T)$ and the time t yield to maturity of a defaultable discount bond with maturity T by $R(r, s, u, t, T)$, the credit spread, $S(r, s, u, t, T)$, is given by

$$\begin{aligned} S(r,s,u,t,T) &= R(r,s,u,t,T) - R^*(r,t,T) \\ &= \frac{1}{T-t}\left[\int_t^T \theta_u D(\tau,T)\,d\tau - C(t,T)\,s + D(t,T)\,u\right]. \end{aligned} \quad (25)$$

It is obvious from Equation (25) that the credit spreads the model implies are independent of the level of the non-defaultable short rate, although the influence of the non-defaultable short rate on the prices of defaultable and non-defaultable bonds is crucial. In addition, the credit spread tends to $\frac{\int_t^T \theta_u D(\tau,T)d\tau}{T-t}$ as s and u tend to 0. $\frac{\int_t^T \theta_u D(\tau,T)d\tau}{T-t}$ can be either positive or negative, depending on the values of the parameters. This is because there is a certain probability that the quality of the firm under consideration will deteriorate (i.e., the uncertainty index will increase) between time t and maturity, although current quality is excellent and therefore the current short rate spread is 0. There will always remain a certain amount of default risk, and the buyer of the defaultable bond should be compensated for it. Nevertheless, if $\theta_u \equiv 0$, zero becomes an absorbing state for u. Then, if s tends to 0, the yield spread will also tend to 0.

Finally, the model is consistent in that the short rate spread implied by the credit spreads equals:[3]

$$\lim_{T \to t} \left[S(r,s,u,t,T)\right] = s(t). \quad (26)$$

Two immediate implications can be drawn from the wider set of parameters influencing the term structure of credit spreads compared to previous models. First, we can expect wider spreads and spreads that are closer to those observed in practice. Second, our model captures more complex properties of credit spreads. To better understand these properties, we start with specific base case parameter values and study the effects of varying them: $s(0) = 0.002$, $u(0) = 0.1$, $\theta_u = 1$, $\hat{a}_u = 1$, $\sigma_u^2 = 0.16$, $b_s = 0.0001$, $\hat{a}_s = 0.1$, $\sigma_s^2 = 0.01$.

Exhibit 1 shows that an increase in b_s has a great impact on the term structure of credit spreads. An increase from 1 basis point to 5 basis points widens the credit spread for a three-year maturity bond from about 120 to about 560 basis points. In general, the larger b_s, the greater the influence of the uncertainty index on the term structure of the short rate spread and the credit spread. The maximum credit spread is observed for a defaultable bond with a maturity of approximately three years.

EXHIBIT 1
Credit Spreads for Different Values of b_s

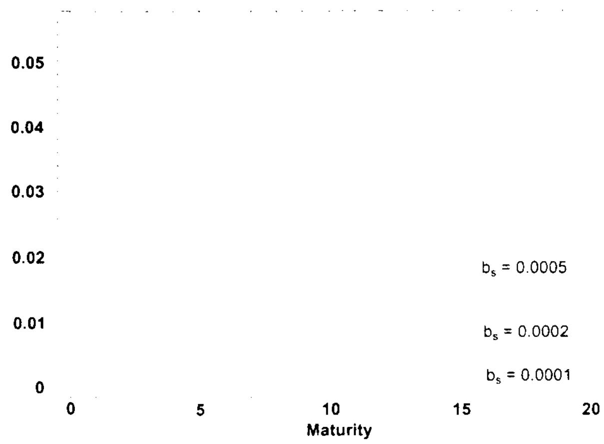
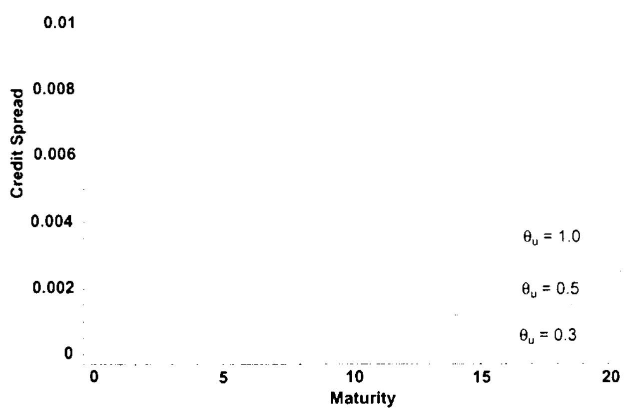


EXHIBIT 2
Credit Spreads for Different Values of θ_u



We can produce a wide range of levels of credit spreads and are not faced with the problem of the traditional Merton model, which cannot reproduce the entire range of spreads observed in the market. All three curves in Exhibit 1 are hump-shaped; the firm faces a great prospect of default during the first years but can overcome its critical financial state over time.

Exhibit 2 shows that the parameter θ_u and therefore the mean-reversion level of the uncertainty index behave as expected; an increase in θ_u widens the credit spread. The parameter represents the long-term expectation of the quality of the firm. The lower this expectation, the narrower the spreads.

Exhibit 3

Credit Spreads for Different Values of $\hat{a}_u$

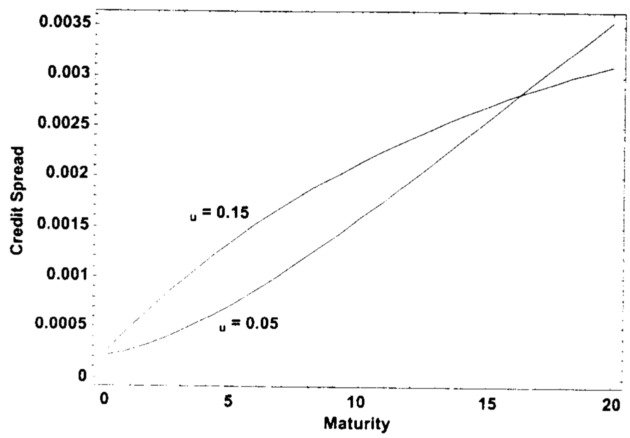


Exhibit 4

Credit Spreads for Different Values of $u(0)$

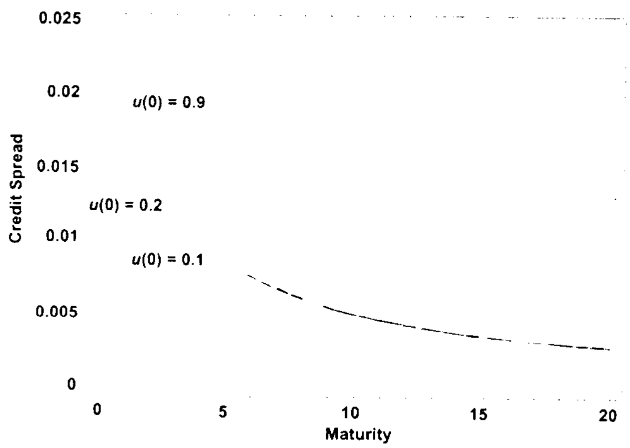


Exhibit 3 shows upward-sloping credit spread term structures. The firm is unlikely to default in the short term but could deteriorate in quality over time. The actual quality of the firm ($u(0) = 0.1$) is much better than the long-run levels of $\frac{\theta_u}{\hat{a}_u} = \frac{1}{0.05} = 20$ or $\frac{\theta_u}{\hat{a}_u} = \frac{1}{0.15} = 6.67$.

The increase of $\hat{a}_u$ from 0.05 to 0.15 has two adverse effects: The credit spread widens in the short and medium term and tightens in the long term. The widening is the effect of the higher speed; the tightening is the effect of the lower long-run level.

Exhibit 4 shows that credit spreads of short-term debt are sensitive to the current quality of the firm. For long-term debt, the long-run level of the quality of the firm is more important than its current quality. In addition, for $u(0) = 0.9$, i.e., for lower-quality firms, we get a downward-sloping term structure of credit spreads. Hence, our model is capable of producing upward-sloping, downward-sloping, and humped term structures of credit spreads, meeting the properties suggested by the empirical results of Sarig and Warga [1989].

Exhibit 5 shows that credit spreads are an increasing function of s. This accords with the fact that the price of a defaultable bond is a decreasing function of s. In contrast with Merton's and many other models, our model produces positive credit spreads even for very short maturities. Thus, our model takes into consideration that instantaneous default is possible. In addition, it captures expected as well as unexpected defaults.

The Recovery Rate

One of the key components of the default process is bondholder recovery in the event of default. We model recovery rates as recoveries of market value. For simplicity, we assume that the stochastic recovery rate w and the default process are independent, and that the recovery rate is paid at maturity (if there is a default prior to maturity). Then we can easily determine an implicit expected recovery rate. These assumptions are not too restrictive, as most models assume constant recovery rates.

By our assumptions, the price of a risky bond is given by

$$\begin{aligned} P(t,T) &= P^*(t,T)\, E^{\mathbf{Q}}\left[w \mathbf{1}_{\{\tau \le T\}} + \mathbf{1}_{\{\tau > T\}} | \mathcal{F}_t\right] \\ &= P^*(t,T)\left[E^{\mathbf{Q}}[w|\mathcal{F}_t]\left(1 - \mathbf{Q}(\tau > T|\mathcal{F}_t)\right) + \mathbf{Q}(\tau > T|\mathcal{F}_t)\right]. \end{aligned} \tag{27}$$

where $P^*(t, T)$ denotes the value at time t of a non-defaultable discount bond assuming the dynamics of the riskless short rate specified by Equation (1). Using Equations (23) and (27), we get

$$E^{\mathbf{Q}}[w(t)|\mathcal{F}_t] = \frac{e^{-\int_t^T \theta_u D(\tau,T)d\tau - C(t,T)s - D(t,T)u} - \mathbf{Q}(\tau > T|\mathcal{F}_t)}{1 - \mathbf{Q}(\tau > T|\mathcal{F}_t)}. \tag{28}$$

Thus, our model yields an alternative way for finding recovery rates if there is limited empirical information available.

Exhibit 5

Credit Spreads for Different Values of $s(0)$

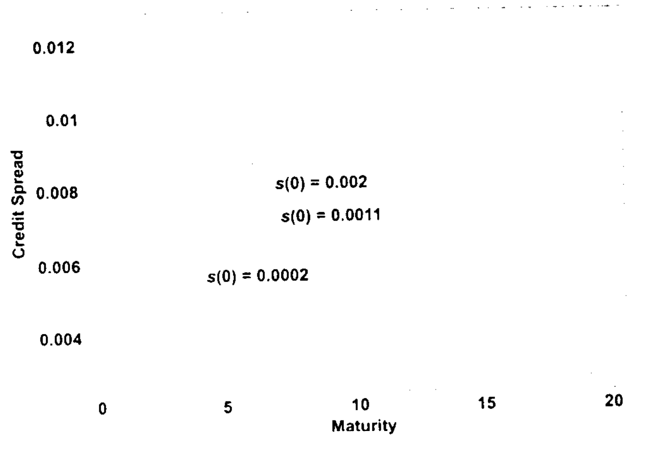


Exhibit 6

Zero-Coupon Default-Free Term Structure

Maturity	Days	Rate
3 days	3	5.01772
1 month	31	4.98284
2 months	62	4.97234
3 months	94	4.96157
6 months	185	4.99058
1 year	367	5.09389
2 years	731	5.79733

Pricing Credit Derivatives

Credit derivatives can be priced using three-dimensional trinomial trees based on our three-factor model. The building method is a modification of the two-stage procedure proposed by Hull and White [1994] for representing a wide range of one-factor interest rate models. We use their procedure to build an interest rate tree corresponding to the short rate model for $\beta = 0$. All processes are assumed to be considered under the equivalent martingale measure $\mathbf{Q}$. As a first step, suppose we must build a trinomial tree for the one-dimensional stochastic process $X(t)$ defined by

$$dX(t) = (\mu(X,t) - a \cdot X(t))\,dt + \sigma dW_*(t), \quad (29)$$

where $W_*(t)$ denotes a one-dimensional Brownian motion; $a > 0$ and $\sigma > 0$ are constants; and $\mu(X,t)$ is a progressively measurable stochastic process that may depend on $X(t)$. Note that $W_*(t)$ may be any of the Brownian motions we consider in our three-factor model, so we use F_t as already defined.

Working with time steps of size Δt, we know that

$$\Delta X(t) = X(t + \Delta t) - X(t) =$$

$$(\mu(X,t) - a \cdot X(t)) \cdot \Delta t + \sigma \cdot \Delta W_*(t) \quad (30)$$

which is normally distributed conditional on F_t with an expected value of

$$E^{Q}[\Delta X(t) \mid F_t] = [\mu(X,t) - aX(t)]\Delta t \quad (31)$$

and a conditional variance of

$$Var^{\mathbf{Q}}[\Delta X(t) \mid \mathcal{F}_t] = \sigma^2 \cdot \Delta t. \quad (32)$$

There are two main conditions for a sequence of discrete-time processes to converge in distribution to the corresponding continuous-time diffusion process (see, e.g., Amin [1995]):

1. The (conditional) mean, variance, and covariance terms per time unit Δt of the discrete sequence of processes must converge to that of the continuous-time diffusion process.
2. The limiting process must have continuous sample paths, i.e., as $\Delta t \to 0$, the maximum of the absolute increment to the process (jump size) at any date must converge to zero.

Since we are dealing with the assumption of uncorrelated Brownian motion, i.e., $\rho_{rs} = \rho_{ru} = \rho_{su} = 0$, these conditions are satisfied for our three-factor model if we ensure the mean and variance conditions for each single one-factor model. The basic building block to do this is to consider the flat stochastic process $X^*(t)$ defined by Equation (29) with $\mu \equiv 0$. To minimize error, we choose a constant step size of $\Delta x^* = \sigma \cdot \sqrt{3 \cdot \Delta t}$ for the distance between the nodes of $X^*(t)$ at each time step (see, e.g., Hull and White [1994]).

If we define (i, j) as the tree node with $t = i\Delta t$ and $X^*(t) = X^*(i\Delta t) = j\Delta x^*$; j_{max} as the smallest integer greater than $\frac{0.184}{a \cdot \Delta t}$, and $j_{min} = -j_{max}$; we can ensure that, for the flat tree, the mean and variance condition holds, and all probabilities are positive if we apply branching methods as follows: standard branching for $j_{min} \leq j \leq j_{max}$ from node (i, j) to nodes $u = (i + 1, j + 1)$, $m = (i + 1, j)$, and $d = (i + 1, j - 1)$ with probabilities

$$p_u = \frac{1}{6} + \frac{a^2j^2\Delta t^2 - aj\Delta t}{2},\ p_m = \frac{2}{3} - a^2j^2\Delta t^2,\ p_d = \frac{1}{6} + \frac{a^2j^2\Delta t^2 + aj\Delta t}{2} \quad (33)$$

upward branching for $j > j_{min}$ and downward branching for $j < j_{max}$.[4] Note that the probabilities at each node depend only on j, a, and Δt, which makes the procedure very robust with respect to the sensitivity analysis of changing yield or spread curves.

In the second building block, we shift each node in the flat tree to ensure that conditions 1 and 2 hold in the resulting tree. Let us therefore define the shift size $\alpha(t)$ at each node by $\alpha(t) = X(t) - X^*(t)$, $t \in [0, T^*]$, which can be described by the differential equation

$$d\alpha(t) = (\mu(X,t) - a \cdot \alpha(t))\,dt \text{ with } \alpha(0) = X(0)\,. \quad (34)$$

The corresponding discrete version is given by

$$\begin{aligned} \Delta\alpha(t) &= \alpha(t+\Delta t) - \alpha(t) = [\mu(X,t) - a\cdot\alpha(t)]\cdot\Delta t \\ &= [\mu(X^* + \alpha, t) - a\cdot\alpha(t)]\cdot\Delta t. \end{aligned} \quad (35)$$

Hence:

$$\alpha(t+\Delta t) = \mu(X^* + \alpha, t)\cdot\Delta t + (1 - a\cdot\Delta t)\cdot\alpha(t)\,. \quad (36)$$

Note that $\alpha(t + \Delta t)$ via $\mu(X, t)$ depends on the predecessor node of the specific node in the tree that has to be shifted. Using Equations (31) with $\mu \equiv 0$ and (35), we get

$$\begin{aligned} E^Q[\Delta X(t)|\mathcal{F}_t] &= E^Q[X^*(t+\Delta t) + \alpha(t+\Delta t) - (X^*(t) + \alpha(t))|\mathcal{F}_t] \\ &= [\mu(X,t) - a\cdot X(t)]\cdot\Delta t. \end{aligned}$$

Similarly, we get

$$Var^Q[\Delta X(t)|\mathcal{F}_t] = Var^Q[\Delta X^*(t)|\mathcal{F}_t] = \sigma^2\cdot\Delta t, \quad (37)$$

which ensures that our sequence of discrete-time processes converges in distribution to the corresponding continuous-time diffusion process.

For the practical application we start at time $t = 0$ and shift the node (0, 0) by $\alpha(0) = X(0)$. Suppose we shift the flat tree up to time $t = i\Delta t$, getting the node values $x_{i,j}$ from $x^*_{i,j}$. Then, the value $x^*_{i+1,j+k}$, $k = -1, 0, 1$, of each node $(i + 1, j + k)$ that is reached by node (i, j) is shifted by

$$\alpha_{i+1,j+k} = \mu(x_{i,j}, t)\Delta t + (1 - a\Delta t)\alpha_{i,j} \quad (38)$$

to the value $x_{i+1,j+k}$. Note that the dependence of $\alpha_{i+1,j+k}$ on $x_{i,j}$ can destroy the recombining property of the flat tree.

The third building block is necessary only for Cox-Ingersoll-Ross processes defined by

$$dY(t) = (\theta(Y,t) - \hat{a}\cdot Y(t))\,dt + \sigma\cdot\sqrt{Y(t)}\,dW(t)\,. \quad (39)$$

Setting $X(t) = G(y, t)$, and using the transformation

$$G(y,t) = 2\cdot\sqrt{y(t)} \quad (40)$$

with Itô's lemma, we get

$$dX(t) = \left[\frac{1}{\sqrt{Y(t)}}\cdot(\theta(Y,t) - \hat{a}\cdot Y(t)) - \frac{1}{4}\cdot\frac{1}{\sqrt{Y(t)}}\cdot\sigma^2\right]dt + \sigma dW(t) \quad (41)$$

Now, since $Y(t) = {}^1\!/_4 X^2(t)$:

$$\begin{aligned} dX(t) &= \left[\frac{2\cdot\theta(\frac{1}{4}\cdot X^2, t) - \frac{1}{2}\cdot\sigma^2}{X(t)} - \frac{\hat{a}}{2}\cdot X(t)\right]dt + \sigma dW(t) \\ &= [\mu(X,t) - a\cdot X(t)]\,dt + \sigma dW(t) \end{aligned} \quad (42)$$

with

$$\mu(X,t) = \frac{2\cdot\theta(\frac{1}{4}\cdot X^2, t) - \frac{1}{2}\cdot\sigma^2}{X(t)} \text{ and } a = \frac{\hat{a}}{2}. \quad (43)$$

We can apply the building blocks to build up a representative trinomial tree for the stochastic process $X(t)$. We get the corresponding tree for $Y(t)$ by setting $Y(t) = {}^1\!/_4 X^2(t)$ in each node of the tree for $X(t)$.

To apply the proposed method for our three-factor model, we start with the uncertainty process

$$du(t) = [\theta_u - \hat{a}_u\cdot u(t)]\,dt + \sigma_u\cdot\sqrt{u(t)}\,dW_u(t)\,. \quad (44)$$

This can be transformed to the process

$$dX_u(t) = \left[\frac{2\cdot\theta_u - \frac{1}{2}\cdot\sigma^2}{X_u(t)} - \frac{\hat{a}_u}{2}\cdot X_u(t)\right]dt + \sigma_u dW_u(t) \quad (45)$$

and

$$\begin{aligned} \alpha_u(t+\Delta t) &= \alpha_u(t) + \left[\frac{2\cdot\theta_u - \frac{1}{2}\cdot\sigma^2}{X_u(t)} - \frac{\hat{a}_u}{2}\cdot\alpha_u(t)\right]\cdot\Delta t \\ &= \frac{2\cdot\theta_u - \frac{1}{2}\cdot\sigma^2}{X_u^*(t) + \alpha_u(t)}\cdot\Delta t + \left(1 - \frac{\hat{a}_u}{2}\cdot\Delta t\right)\cdot\alpha_u(t) \end{aligned} \quad (46)$$

with $\alpha_u(0) = X_u(0) = 2\sqrt{u(0)}$.

Similarly, given $u(t)$ at time t, for the spread process:

$$ds(t) = [b_s \cdot u(t) - \hat{a}_s \cdot s(t)]\, dt + \sigma_s \cdot \sqrt{s(t)}\, d\hat{W}_s(t) \tag{47}$$

we get the transformations

$$dX_s(t) = \left[\frac{2 \cdot b_s \cdot u(t) - \frac{1}{2} \cdot \sigma_s^2}{X_s(t)} - \frac{\hat{a}_s}{2} \cdot X_s(t)\right] dt + \sigma_s d\hat{W}_s(t) \tag{48}$$

and

$$\alpha_s(t + \Delta t) = \frac{2 \cdot b_s \cdot u(t) - \frac{1}{2} \cdot \sigma_s^2}{X_s^*(t) + \alpha_s(t)} \cdot \Delta t + \left(1 - \frac{\hat{a}_s}{2} \cdot \Delta t\right) \cdot \alpha_s(t) \tag{49}$$

with $\alpha(0) = X_s(0) = 2 \cdot \sqrt{s(0)}$.

The short rate process is given by

$$dr(t) = [\theta_r(t) - \hat{a}_r \cdot r(t)]\, dt + \sigma_s \cdot r^{\beta}(t)\, d\hat{W}_r(t) \tag{50}$$

with $\beta \in \{0, 1/2\}$. For $\beta = 1/2$ we have to transform the short rate process to

$$dX_r(t) = \left[\frac{2 \cdot \theta(t) - \frac{1}{2} \cdot \sigma_r^2}{X_r(t)} - \frac{\hat{a}_r}{2} \cdot X_r(t)\right] dt + \sigma_r d\hat{W}_r(t). \tag{51}$$

For $\beta = 0$ the transformation is not necessary, and we may consider

$$dX_r(t) = [\theta(t) - \hat{a}_r \cdot X_r(t)]\, dt + \sigma_r d\hat{W}_r(t), \tag{52}$$

i.e., $X(t) = r(t)$.

The corresponding α values are given by

$$\alpha_r(t + \Delta t) = \frac{2 \cdot \theta(t) - \frac{1}{2} \cdot \sigma_r^2}{X_r^*(t) + \alpha_r(t)} \cdot \Delta t + \left(1 - \frac{\hat{a}_r}{2} \cdot \Delta t\right) \cdot \alpha_r(t) \tag{53}$$

with $\alpha_r(0) = X_r(0) = 2\sqrt{r(0)}$ for $\beta = 1/2$, and

$$\alpha_r(t + \Delta t) = \theta(t) \cdot \Delta t + (1 - \hat{a}_r \cdot \Delta t) \cdot \alpha_r(t) \tag{54}$$

with $\alpha_r(0) = X_r(0) = r(0)$ for $\beta = 0$.

Note that the short rate model can be fitted to the yield curve as $\theta(t)$ is time-dependent. Also note that for the Hull-White model ($\beta = 0$), the shift size $\alpha_r(t)$ is the same for all nodes (i, j) at time $t = i \cdot \Delta t$. To build an interest rate tree for this case, Hull and White [1994] propose a slightly different method that they claim to be numerically more efficient.

We are now able to build a three-dimensional trinomial tree for the process $[r(t), s(t), u(t)]$. In contrast to the trinomial trees of Chen [1996], Amin [1995], or Boyle [1988], the branching as well as the probabilities do not change with a changing drift that makes the tree more efficient, especially for risk management purposes. The probabilities for each node in the three-dimensional tree are given by the product of the one-dimensional processes. Note that although we have to consider all possible combinations (r_{ij}, s_{ij}, u_{ij}) at time $t = i \cdot \Delta t$ and nodes $j_r \cdot \Delta r$, $j_s \cdot \Delta s$, and $j_u \cdot \Delta u$, the total number is limited by the branching method.

By this three-dimensional tree method, we are able to price options on defaultable bonds as well as a great variety of credit derivatives. Suppose we want to price credit spread options predicated on the base case. Credit spread options allow investors whose portfolio values are highly sensitive to shifts in the spread between defaultable and non-defaultable yields to manage and hedge their exposure to this type of risk. They convey the right to sell a defaultable discount bond with maturity T at a fixed strike spread K over a default-free reference yield $R^*(r, \tau, T)$ at maturity $\tau \leq T$. The option expires worthless if K exceeds the actual market spread at time τ.

We assume that the issuer of the spread option is default-free. Let $C(r, s, u, t, T; \tau, K)$ denote the time t value of this spread option. Then, at the expiration date τ:

$$C(r, s, u, \tau, T; \tau, K) = [\exp[-(T - \tau)(K + R^*(r, \tau, T))] - P(r, s, u, \tau, T)]_+ \tag{55}$$

The pricing of the spread option consists of two main steps:

1. Building the three-dimensional tree as described.
2. Determining the fair value of the spread option at time t using a backward recursive procedure as usual:
   a. Calculating the values of the option at all maturity τ tree nodes by using the payoff function (55).
   b. Calculating the values of the option at all nodes one time step prior to maturity from the option values at maturity, and so on.

As an example of the model implementation, we assume the default-free zero curve in Exhibit 6, all rates continuously compounded. Linear interpolation is used to generate data points for maturities between those indicated. Then, using $\Delta t = 1/2$; $T = 2$; $\tau = 1$; $t = 0$; $\hat{a}_r = 1.05 \cdot 10^{-8}$;

$\sigma_r = 0.01$; $K = 40$ basis points; $u(0) = 0.1$; $\theta_u = 1$; $\hat{a}_u = 1$; $\sigma_u^2 = 0.16$; $s(0) = 0.02$; $b_s = 0.0001$; $\hat{a}_s = 0.1$; and $\sigma_s^2 = 0.01$ yields

$$C(r, s, u, t, T; \tau, K) = 0.726447\% \tag{56}$$

Note that the initial default-free interest rate value of the tree is implied by the assumed zero curve as 4.99058%. Similarly, the initial spread value of the tree is implied by the assumed short rate spread of $s(0) = 0.02$ as 2.64903%.

III. FITTING THE MODEL TO MARKET DATA

The ultimate success or failure of pricing formulas ends on collecting the necessary information for determining model parameter values. We suggest one way to find meaningful values for the parameters of the three processes r, s, and u.

The Uncertainty Index

Many practitioners use ratings of rating agencies as a proxy for the quality of a firm. Usually the better the rating, the lower the probability of default. We assume that there are six rating classes, which we denote by 1, 2, . . . , 6. Rating class 6 corresponds to default; rating class 1 consists of the instruments with the highest quality (i.e., the lowest uncertainty) of defaultable bonds. Each firm (or, more specifically, each debt instrument) belongs to exactly one rating class according to its quality or its "uncertainty" given by the uncertainty index of the firm.

There is a one-to-one mapping from the value of the uncertainty index onto the set of rating classes, i.e., there are thresholds ξ_j, $j = 1, \ldots, 5$, such that a firm is in default if $u > \xi_5$, has rating j if $\xi_{j-1} < u \le \xi_j$, $j = 2, \ldots, 5$, or has rating 1 if $u \le \xi_1$. Using the non-central χ^2 distribution property of $u(t)$, the unknown parameters of the process u and the unknown thresholds can be expressed by the transition probabilities of the firm (see Lamberton and Lapeyre [1996, p. 129 ff.]).

If we suppose that the transition probabilities are given for the time interval $[0, t]$, the equations as follows hold:

$$\mathbf{P}(u(t) \le \xi_1 | u(0)) = \mathbf{F}_{\frac{4\theta_u}{\sigma_u^2}, \varsigma}\left(\frac{\xi_1}{L}\right) \tag{57}$$

$$\mathbf{P}(u(t) \le \xi_1 | u(0)) + \sum_{k=2}^{j} \mathbf{P}(\xi_{k-1} < u(t) \le \xi_k | u(0)) = \mathbf{F}_{\frac{4\theta_u}{\sigma_u^2}, \varsigma}\left(\frac{\xi_j}{L}\right) \tag{58}$$

$$\mathbf{P}(u(t) > \xi_5 | u(0)) = 1 - \mathbf{F}_{\frac{4\theta_u}{\sigma_u^2}, \varsigma}\left(\frac{\xi_5}{L}\right) \tag{59}$$

where $L = \frac{\sigma_u^2}{4a_u}\left(1 - e^{-a_u t}\right)$, $\varsigma = \frac{4u(0)a_u}{\sigma_u^2(e^{a_u t} - 1)}$ and $\mathbf{F}_{\frac{4\theta_u}{\sigma_u^2}, \varsigma}$ denotes the distribution function of the non-central chi-squared law with $\frac{4\theta_u}{\sigma_u^2}$ degrees of freedom and parameter ς. The transition probabilities are the same within each rating class.

To embed this discrete setting in our continuous framework, we assume that for each possible initial rating the rating agencies' probabilities correspond to a certain initial uncertainty process value within the range of admissible values; i.e., rating 1 corresponds to some uncertainty process value $u_1(0) = \alpha_1 \xi_1$ for some $0 \le \alpha_1 \le 1$, and rating k, $2 \le k \le 5$, corresponds to some uncertainty process value $u_k(0) = (1 - \alpha_k)\,\xi_{k-1} + \alpha_k \xi_k$ for some $0 \le \alpha_k \le 1$.

If we set up Equations (57) and (58) for each rating class (besides rating class 6), we get a system of 25 equations with 25 unknown variables that we must solve under the constraints $\xi_1 < \xi_2 < \xi_3 < \xi_4 < \xi_5$ and $0 \le \alpha_k \le 1$, $1 \le k \le 5$. One possibility is to apply the multidimensional downhill simplex method, which was introduced by Nelder and Mead [1965]. Because we use real-world transition probabilities, we can estimate only a_u but not $\hat{a}_u$ (although for pricing purposes we are more interested in $\hat{a}_u$). Therefore, we estimate λ_u together with the parameters for the short rate spread from market prices of observed credit spreads.

The Non-Defaultable Short Rate

The most intuitive procedure to fit the non-defaultable short rate to market prices (such as broker quotes) is to choose the values for $\hat{a}_r$ and σ_r that best fit a series of non-defaultable interest rate options such as caps. The parameter σ_r determines the overall volatility of the non-defaultable short rate. The parameter $\hat{a}_r$ determines the relative volatilities of long and short rates.

The procedure is to choose the values of $\hat{a}_r$ and σ_r that minimize

$$\sum_i (P_i^*(\hat{a}_r, \sigma_r) - V_i)^2. \tag{60}$$

where P_i^* is the price given by the model for the i-th interest rate option, and V_i is the market price of the i-th interest rate option. The procedure is quite easy for the Hull/White model, where cap prices can be calculated analytically. Again the problem can be solved by applying the multidimensional downhill simplex method.

The Short Rate Spread

The remaining parameters to be estimated are b_s, $\hat{a}_s$, σ_s, and λ_u. We choose the values for these parameters that best fit a series of observed credit spreads. As with the estimation for the parameters of the non-defaultable short rate, the procedure is to choose the values of b_s, $\hat{a}_s$, σ_s, and λ_u that minimize

$$\sum_i (S_i(b_s, \hat{a}_s, \sigma_s, \lambda_u) - Y_i)^2. \tag{61}$$

where S_i is the model value of the i-th credit spread and Y_i is its observed market value. Again, for the optimization procedure we can use a multidimensional downhill simplex method.

Example

Suppose we apply our theoretical framework and our calibration methods to set up a firm-specific model for General Motors. We use data from Reuters, Bloomberg, and S&P to estimate the model parameters:

1. The non-defaultable short rate:
   Using the U.S. Cap-Volatilities (May 4, 1999) from Reuters (1 year: 11.25%, 2 years: 15.25%), we get the estimations $\hat{a}_r = 1.05 \cdot 10^{-8}$, and $\sigma_r = 1.01949464\%$.
2. The uncertainty index:
   We assume that $\alpha_k = \frac{1}{2} \forall k$ (i.e., ratings correspond to the average quality within a rating class) and that there are five rating classes.

   The average one-year transition rates by S&P (July 1998) are given in Exhibit 7, Panel A. We modify the matrix by aggregating the rating classes AAA and AA in rating class 1, BBB and BB in rating class 3 and B and CCC in rating class 4. A is rating class 2, and D is rating class 5. The modified transition matrix is given in Panel B of Exhibit 7.

EXHIBIT 7

Panel A. —Transition Rates S&P

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	90.82	8.26	0.74	0.06	0.11	0.00	0.00	0.00
AA	0.65	90.88	7.69	0.58	0.05	0.13	0.02	0.00
A	0.08	2.42	91.30	5.23	0.68	0.23	0.01	0.05
BBB	0.03	0.31	5.87	87.45	4.96	1.08	0.12	0.18
BB	0.02	0.12	0.64	7.70	81.08	8.39	0.98	1.08
B	0.00	0.10	0.24	0.44	6.81	82.90	3.90	5.60
CCC	0.20	0.00	0.40	1.19	2.56	11.24	61.97	22.50

Panel B. Modified Rates

	1	2	3	4	5
1	95.31	4.22	0.40	0.07	0.00
2	2.50	91.30	5.91	0.24	0.05
3	0.24	3.26	90.59	5.28	0.63
4	0.15	0.30	5.50	80.00	14.05

In S&P's notation, General Motors' rating is A, which corresponds to our rating class 2. Using the downhill simplex algorithm, we get the parameter estimates for this rating class:

$\theta_u = 0.0653562131910263$,
$\sigma_u = 0.09999999994330677$,
$a_u = 0.13499953576965013$;
and the threshold values
$\xi_1 = 0.0436501349994976$,
$\xi_2 = 0.2540403487950756$,
$\xi_3 = 0.6491354397214147$, and
$\xi_4 = 1.1806484698650033$.

3. The short rate spread:
   For estimating the remaining parameters we use U.S. dollar fixed-rate bond data (semiannual coupons, S&P rating class A of General Motors, which corresponds to our rating class 2) from Bloomberg (May 4, 1999):
   a. 9-1/8, 07/18/00, Clean Price: 104.15, Yield: 5.497.
   b. 9.02, 06/07/01, Clean Price: 106.32, Yield: 5.758.
   c. 8-7/8, 06/11/01, Clean Price: 106.07, Yield: 5.760.
   d. 9.2, 07/02/01, Clean Price: 106.86, Yield: 5.769.

Using $s(0) = 80$ basis points, the downhill simplex algorithm yields the estimates: $b_s = 0.0000997$, $\hat{a}_s = 0.328893$, $\sigma_s = 0.2$, and $\lambda_u = 0.19$.

IV. SUMMARY

Our new model for pricing defaultable bonds and other default-related instruments captures typical properties of defaultable and non-defaultable interest rates. While diffusion-based models like Merton's model generate unrealistically low short-maturity spreads, our model can generate any spreads observed in reality. Although intensity-based models can correct this, these approaches lose too much structural information, making the models less relevant for changing economic conditions. Hence, - model overcomes many weaknesses of the structural as well as the intensity-based models and combines their strengths. The methodology can be used on an issuer basis as well as in a rating-based framework.

Future research will consider alternative calibration methods and intensive empirical studies. Applying the framework to emerging market bonds should be particularly interesting.

APPENDIX

Proof of Theorem

According to the lemma, the value at time t, $P(r, s, u, t, T)$, of a defaultable discount bond is the solution to the PDE

$$\begin{aligned} 0 \;=\; & \frac{1}{2}\sigma_r^2 r^{2\beta} P_{rr} + \frac{1}{2}\sigma_s^2 s P_{ss} + \frac{1}{2}\sigma_u^2 u P_{uu} \\ & + (\theta_r - \hat{a}_r r) P_r + (b_s u - \hat{a}_s s) P_s \\ & + (\theta_u - \hat{a}_u u) P_u + P_t - (r + s) P. \end{aligned} \tag{A-1}$$

with boundary condition $P(r, s, u, t, T) = 1$.

If $\beta = 0$, plugging in the partial derivatives of P yields the system of ordinary differential equations:

$$\hat{a}_r B - B_t - 1 = 0 \tag{A-2}$$

$$\frac{1}{2}\sigma_s^2 C^2 + \hat{a}_s C - C_t - 1 = 0 \tag{A-3}$$

$$\frac{1}{2}\sigma_u^2 D^2 + \hat{a}_u D - D_t - b_s C = 0 \tag{A-4}$$

$$A\left(\theta_r B + \theta_u D - \frac{1}{2}\sigma_r^2 B^2\right) - A_t = 0. \tag{A-5}$$

Equation (A-3) has Ricatti form $C_t = \frac{1}{2}\sigma_s^2 C^2 + \hat{a}_s C - 1$. The solution is

$$C(t,T) = \frac{-2w'(t,T)}{\sigma_s^2 w(t,T)} \tag{A-6}$$

where $w(t, T)$ satisfies

$$w'' - \hat{a}_s w' - \frac{1}{2}\sigma_s^2 w = 0. \tag{A-7}$$

From $\hat{a}_s^2 + 2\sigma_s^2 > 0$ it follows that all solutions of the ODE (A-7) are given by $w = \alpha_1 e^{\kappa_1^{(s)} t} + \alpha_2 e^{\kappa_2^{(s)} t}$, where α_1 and α_2 are some constants, and $\kappa_1^{(s)}$ and $\kappa_2^{(s)}$ are given by

$$\kappa_{1/2}^{(x)} = \frac{\hat{a}_x}{2} \pm \frac{1}{2}\sqrt{\hat{a}_x^2 + 2\sigma_x^2}. \tag{A-8}$$

Then all solutions to Equation (A-3) are given by

$$C(t,T) = -\frac{2}{\sigma_s^2}\frac{\alpha_1 \kappa_1^{(s)} e^{\kappa_1^{(s)} t} + \alpha_2 \kappa_2^{(s)} e^{\kappa_2^{(s)} t}}{\alpha_1 e^{\kappa_1^{(s)} t} - \alpha_2 e^{\kappa_2^{(s)} t}}. \tag{A-9}$$

In order to satisfy the boundary condition $C(T, T) = 0$, the constant α_1 equals

$$\alpha_1 = -\alpha_2 \frac{\kappa_2^{(s)}}{\kappa_1^{(s)}} e^{\left(\kappa_2^{(s)} - \kappa_1^{(s)}\right) T}. \tag{A-10}$$

and therefore

$$C(t,T) = \frac{1 - e^{-\delta_s (T-t)}}{\kappa_1^{(s)} - \kappa_2^{(s)} e^{-\delta_s (T-t)}}. \tag{A-11}$$

where

$$\delta_x = \sqrt{\hat{a}_x^2 + 2\sigma_x^2}. \tag{A-12}$$

Equation (A-4) has Ricatti form $D_t = \frac{1}{2}\sigma_u^2 D^2 + \hat{a}_u D - b_s C$. The solution is

$$D(t,T) = \frac{-2v'(t,T)}{\sigma_u^2 v(t,T)}. \tag{A-13}$$

where $v(t, T)$ satisfies

$$v'' - \hat{a}_u v' - \frac{1}{2} b_s \sigma_u^2 C v = 0. \tag{A-14}$$

The solutions to Equation (A-14) are given by

$$v(t,T) = \vartheta_1 \left(\sigma_u^2 e^{-\delta_s(T-t)}\right)^{\frac{\hat{a}_u}{2\delta_s}-\phi\left(\kappa_1^{(s)}\right)} F_1(t,T) + \vartheta_2 \left(\sigma_u^2 e^{-\delta_s(T-t)}\right)^{\frac{\hat{a}_u}{2\delta_s}+\phi\left(\kappa_1^{(s)}\right)} F_3(t,T), \quad \text{(A-15)}$$

where ϑ_1 and ϑ_2 are some constants, and

$$F_1(t,T) = F\left(-\phi\left(\kappa_1^{(s)}\right)-\phi\left(\kappa_2^{(s)}\right), -\phi\left(\kappa_1^{(s)}\right)+\phi\left(\kappa_2^{(s)}\right), 1-2\phi\left(\kappa_1^{(s)}\right), \kappa_2^{(s)}/\kappa_1^{(s)} e^{-\delta_s(T-t)}\right). \quad \text{(A-16)}$$

$$F_3(t,T) = F\left(\phi\left(\kappa_1^{(s)}\right)-\phi\left(\kappa_2^{(s)}\right), \phi\left(\kappa_1^{(s)}\right)+\phi\left(\kappa_2^{(s)}\right), 1+2\phi\left(\kappa_1^{(s)}\right), \kappa_2^{(s)}/\kappa_1^{(s)} e^{-\delta_s(T-t)}\right). \quad \text{(A-17)}$$

with

$$\phi(g) = \frac{\sqrt{\hat{a}_u^2 g + 2b_s\sigma_u^2}}{2\delta_s\sqrt{g}} \quad \text{(A-18)}$$

and $F(a, b, c, z)$ is the hypergeometric function.[5]

Differentiation of $v(t, T)$ yields

$$v'(t,T) = y_1(t,T) + y_2(t,T), \quad \text{(A-19)}$$

where

$$y_1(t,T) = \vartheta_1 \left(\sigma_u^2 e^{-\delta_s(T-t)}\right)^{\frac{\hat{a}_u}{2\delta_s}-\phi\left(\kappa_1^{(s)}\right)} \cdot \left(\left(\frac{\hat{a}_u}{2}-\delta_s\phi\left(\kappa_1^{(s)}\right)\right) F_1(t,T) - \zeta_1 e^{-\delta_s(T-t)} F_2(t,T)\right), \quad \text{(A-20)}$$

$$y_2(t,T) = \vartheta_2 \left(\sigma_u^2 e^{-\delta_s(T-t)}\right)^{\frac{\hat{a}_u}{2\delta_s}+\phi\left(\kappa_1^{(s)}\right)} \cdot \left(\left(\frac{\hat{a}_u}{2}+\delta_s\phi\left(\kappa_1^{(s)}\right)\right) F_3(t,T) - \zeta_2 e^{-\delta_s(T-t)} F_4(t,T)\right), \quad \text{(A-21)}$$

with

$$F_2(t,T) = F\left(1-\phi\left(\kappa_1^{(s)}\right)-\phi\left(\kappa_2^{(s)}\right), 1-\phi\left(\kappa_1^{(s)}\right)+\phi\left(\kappa_2^{(s)}\right), 2-2\phi\left(\kappa_1^{(s)}\right), \kappa_2^{(s)}/\kappa_1^{(s)} e^{-\delta_s(T-t)}\right). \quad \text{(A-22)}$$

$$F_4(t,T) = F\left(1+\phi\left(\kappa_1^{(s)}\right)-\phi\left(\kappa_2^{(s)}\right), 1+\phi\left(\kappa_1^{(s)}\right)+\phi\left(\kappa_2^{(s)}\right), 2+2\phi\left(\kappa_1^{(s)}\right), \kappa_2^{(s)}/\kappa_1^{(s)} e^{-\delta_s(T-t)}\right). \quad \text{(A-23)}$$

and

$$\zeta_{1/2} = \delta_s \frac{\kappa_2^{(s)}}{\kappa_1^{(s)}} \frac{\phi^2\left(\kappa_2^{(s)}\right)-\phi^2\left(\kappa_1^{(s)}\right)}{1 \mp 2\phi\left(\kappa_1^{(s)}\right)}. \quad \text{(A-24)}$$

In order to satisfy the boundary condition $D(T, T) = 0$, ϑ_1 equals

$$\vartheta_1 = \vartheta_2 \left(\sigma_u^2\right)^{2\phi(\kappa_1(s))} \frac{\varphi_1}{\varphi_2}. \quad \text{(A-25)}$$

where $\varphi_1 = \zeta_2 F_4(T,T) - \xi_1 F_3(T,T)$,

$\varphi_2 = -\zeta_1 F_2(T,T) + \xi_2 F_1(T,T)$, $\xi_1 = \left(\frac{\hat{a}_u}{2}+\delta_s\phi\left(\kappa_1^{(s)}\right)\right)$,

and $\xi_2 = \left(\frac{\hat{a}_u}{2}-\delta_s\phi\left(\kappa_1^{(s)}\right)\right)$. From Equations (A-13), (A-15), (A-19), and (A-25), $D(t, T)$ can be determined as

$$D(t,T) = \frac{-2\varphi_1\left(\xi_2 F_1(t,T) - \zeta_1 e^{-\delta_s(T-t)} F_2(t,T)\right)}{\sigma_u^2\left[\varphi_1 F_1(t,T) + \varphi_2 e^{-2\delta_s\phi\left(\kappa_1^{(s)}\right)(T-t)} F_3(t,T)\right]} - \frac{2\varphi_2 e^{-2\delta_s\phi\left(\kappa_1^{(s)}\right)(T-t)}\left(\xi_1 F_3(t,T) - \zeta_2 e^{-\delta_s(T-t)} F_4(t,T)\right)}{\sigma_u^2\left[\varphi_1 F_1(t,T) + \varphi_2 e^{-2\delta_s\phi\left(\kappa_1^{(s)}\right)(T-t)} F_3(t,T)\right]}. \quad \text{(A-26)}$$

Consider the case $\beta = 0$:

The solution to Equation (A-2) that satisfies the boundary condition $B(T, T) = 0$ is:

$$B(t,T) = \frac{1}{\hat{a}_r}\left[1 - e^{-\hat{a}_r(T-t)}\right]. \quad \text{(A-27)}$$

By direct substitution, the solution to Equation (A-5) for A that satisfies the boundary condition $A(T, T) = 1$ is

$$A(t,T) = \exp\left[-\int_t^T \left(\theta_r(\tau) B(\tau,T) + \theta_u D(\tau,T) - \frac{1}{2}\sigma_r^2 B^2(\tau,T)\right) d\tau\right]. \quad \text{(A-28)}$$

According to Equation (15):

$$-\int_t^T \left(\theta_r(\tau) B(\tau,T) - \frac{1}{2}\sigma_r^2 B^2(\tau,T)\right) d\tau = \ln A^*(t,T), \quad \text{(A-29)}$$

and, in addition, according to Equation (A-13):

$$\theta_u \int_t^T D(\tau,T)\, d\tau = \frac{-2\theta_u}{\sigma_u^2}\left[\ln\left|\frac{v(T,T)}{v(t,T)}\right|\right], \quad \text{(A-30)}$$

where $\vartheta_1 = \vartheta_2 \left(\sigma_u^2\right)^{2\phi(\kappa_1(s))} \frac{\varphi_1}{\varphi_2}$.[6]

ENDNOTES

[1]Assumptions include no taxes and transaction costs, perfectly divisible assets, no borrowing-lending spread, continuous costless trading and short-selling, and equal access to information for all investors.

[2]Note that we implicitly assume that (compare to Chen [1996], p. 5):

$$\gamma_r(t) = \lambda_r \sigma_r r(t)^{1-\beta}$$

$$\gamma_s(t) = \lambda_s \sigma_s \sqrt{s(t)}$$

and

$$\gamma_u(t) = \lambda_u \sigma_u \sqrt{u(t)}, \; 0 \le t \le T^*,$$

where λ_r, λ_s, and λ_u are real-valued constants and Novikov's ndition is satisfied.

[3] $$\lim_{T \to t} \frac{\int_t^T \theta_u D(\tau, T) d\tau}{T-t} = \theta_u D(T, T) = 0$$

$$\lim_{T \to t} \frac{C(t,T)}{T-t} = \lim_{T \to t} \frac{\left(1 - e^{-\delta_s (T-t)}\right) s(t)}{\left(\kappa_1^{(s)} - \kappa_2^{(s)} e^{-\delta_s (T-t)}\right)(T-t)} = s(t)$$

$$\lim_{T \to t} \frac{D(t,T) u(t)}{T-t} = \lim_{T \to t} \frac{-2v'(t,T) u(t)}{\sigma_u^2 v(t,T)(T-t)} = ($$

as

$$v''(t,t) = -\tfrac{1}{2}\sigma_u^2 \left(\hat{a}_u D(t,t) - b_s C(t,t)\right) v(t,t) = 0$$

and $v(t, t) \neq 0$.

[4]Upward branching for $j < j_{min}$ from node (i, j) to nodes $d = (i+1, j+1)$, and $u = (i+1, j+2)$, with probabilities:

$$p_u = \frac{1}{6} + \frac{a^2 j^2 \Delta t^2 + aj\Delta t}{2}$$

$$p_m = -\frac{1}{3} - a^2 j^2 \Delta t^2 - 2aj\Delta t$$

$$p_d = \frac{7}{6} + \frac{a^2 j^2 \Delta t^2 + 3aj\Delta t}{2}.$$

Downward branching for $j > j_{max}$ from node (i, j) to nodes $d = (i+1, j-2)$, $m = (i+1, j-1)$, and $u = (i+1, j)$ with probabilities:

$$p_u = \frac{7}{6} + \frac{a^2 j^2 \Delta t^2 - 3aj\Delta t}{2}$$

$$p_m = -\frac{1}{3} - a^2 j^2 \Delta t^2 + 2aj\Delta t$$

$$p_d = \frac{1}{6} + \frac{a^2 j^2 \Delta t^2 - aj\Delta t}{2}.$$

[5]The hypergeometric function, usually denoted by F, has series expansion

$$F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

where $(a)_0 = 1$, $(a)_n = a(a+1)(a+2) \ldots (a+n-1)$, and $n \in N$, and is the solution of the hypergeometric differential equation

$$z(1-z)y'' + [c - (a+b+1)z]y' - aby = 0$$

The hypergeometric function can be written as an integral

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt, \; (c > b > 0)$$

and is also known as the Gauss series or the Kummer series.

[6]If $\beta = 1/2$, plugging in the partial derivatives of P yields the system of ODEs:

$$\frac{1}{2}\sigma_r^2 B^2 + \hat{a}_r B - B_t - 1 = 0$$

$$\frac{1}{2}\sigma_s^2 C^2 + \hat{a}_s C - C_t - 1 = 0$$

$$\frac{1}{2}\sigma_u^2 D^2 + \hat{a}_u D - D_t - b_s C = 0$$

$$A(\theta_r B + \theta_u D) - A_t = 0.$$

The solution for C and D are the same as in the case $\beta = 0$. As in Equation (A-11), the solution to the first equation above that satisfies the boundary condition $B(T, T) = 0$ is

$$B(t,T) = \frac{1 - e^{-\delta_r (T-t)}}{\kappa_1^{(r)} - \kappa_2^{(r)} e^{-\delta_r (T-t)}},$$

where $\kappa_{1/2}^{(r)}$ is defined as in Equation (A-8) and δ_r is given by Equation (A-12).

By direct substitution, the solution to the last equation for A that satisfies the boundary condition $A(T, T) = 1$ is

$$A(t,T) = A(0,T) \exp\left[\int_0^t \theta_r(\tau) B(\tau,T) + \theta_u D(\tau,T) d\tau\right].$$

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