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Portfolio Optimization Under Credit Risk

Summary

Based on the models of Hull & White (1990) for the pricing of non-defaultable bonds and Schmid & Zagst (2000) for the pricing of defaultable bonds we develop a framework for the optimal allocation of assets out of a universe of sovereign bonds with different time to maturity and quality of the issuer. We estimate the model parameters by applying Kalman filtering methods as described in (Schmid & Kalemanova 2002). Based on these estimates we simulate the prices for a given set of bonds for a future time horizon. For each future time step and for each given portfolio composition these scenarios yield distributions of future cash flows and portfolio values. We show how the portfolio composition can be optimized by maximizing the expected final value or return of the portfolio under given constraints.

Keywords: Portfolio Optimization, Defaultable Bonds

1 Introduction[1]

The process of performing an optimal asset allocation basically deals with the problem of finding a portfolio that maximizes the expected utility of the portfolio manager. As it is done throughout the traditional portfolio theory introduced by Markowitz (1952) and Sharpe (1964), the problem of finding an expected utility maximizing portfolio for a risk averse portfolio manager, represented by a concave utility function, can be restricted to finding an optimal combination of the two parameters mean and variance. This dramatically simplifies the whole asset allocation process and is known as mean-variance analysis. It is the aim of the portfolio manager to find a portfolio that maximizes his expected return under a given risk level or a portfolio that minimizes his risk under a given return level. Risk in this case is measured by the variance of the portfolio return. Unfortunately, selection rules based on the two parameters mean and variance are of limited generality as they are optimal only if the utility function is quadratic or the return distribution is normal. Furthermore, the use of variance as a measure of risk for asymmetric return distributions has already been questioned by financial theorists like, e.g., Markowitz (1991), p. 188–201. He states that portfolio analyses based on semi-variance tend to produce better portfolios than those based on variance. Unfortunately, the computing cost involved with the use of semi-variance is much higher than that with the analyses based on variance as we need the entire joint distribution for the first technique while we only need the covariance matrix for the latter.

Extensive research has been done to derive concepts for ordering uncertain prospects resulting in principles like the stochastic dominance of order 1, 2 or 3 (see, e.g., (Bawa 1975) or (Martin, Cox & MacMinn 1988)). Bawa argues that it is quite reasonable to assume that the average investor decides consistent with a finite, increasing and concave utility function with decreasing absolute risk aversion, and that there is a strong rationale for using meanlower partial variance rules as an approximation for the portfolio selection of this kind of investor. Under this rule one portfolio is better than the other if its mean is not lower and its lower partial variance is not higher than that of the other portfolio at every possible benchmark. As the lower partial variance is a generalization of the semi-variance, this follows the suggestion of Markowitz, still leaving the computational problems, because we have to calculate the lower partial variances for each distribution at every possible benchmark. If we can not assume special distributions to simplify that problem, the solution is often approximated by optimizing with a finite set of representative benchmarks.

The so-called lower partial moments can be considered as a generalization of the lower partial variance (see, e.g. (Harlow & Rao 1989) or (Harlow 1991)) and are therefore consistent with the previous concept. Because they measure the risk of falling below a given benchmark, they have become very popular measures for the risk of a portfolio, especially when the return distribution may be asymmetric. Consequently, we will use lower partial moments to control the downside risk of the portfolio exposure in this paper. Because we do not have any information on the distribution of the possible or optimal portfolios we approximate the distribution function based on a simulation model. The assets we allow for an allocation are sovereign bonds with different time to maturity and quality of the issuer. Some of these bonds may default and thus stop paying interest. Or the value of a bond may fall only because the rating of the issuing country has changed.

The risk of price changes with respect to changes in the rating of a country or company is called credit or default risk. For modelling default risk we use the approach of Schmid & Zagst (2000). They allow for a Hull-White or a generalized CIR process to describe the short rate r. The quality of the firm is modelled by a CIR type of uncertainty or signaling process u which also has an influence on the explicitely modelled mean-reverting short rate spread s. They derive

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a closed-form solution for defaultable zero coupon bonds and show how credit derivatives can be priced using a tree-based method.

The paper is organized as follows. In section 2 we describe the model of Schmid and Zagst for pricing defaultable bonds. In section 3 we discuss the empirical data set we use and show how the model parameters can be estimated by applying Kalman filtering methods as described in (Schmid & Kalemanova 2002). Based on these estimates we simulate the prices of the bonds for a future time horizon. For each future time step and for each given portfolio composition these scenarios yield distributions of future cash flows and portfolio values. In section 4 we show how the portfolio composition can be optimized by maximizing the expected final value or return of the portfolio under a given set of constraints. One set of restrictions is due to a minimum required cash flow per period to cover the liabilities of a company, the second set limits the tolerated risk. As discussed above, both risks are measured by lower partial moments to account for the downside potential we have to consider. To visualize our methodology we present a case study for a portfolio consisting of German, Italian, and Greek sovereign bonds in section 5.

2 The Underlying Term Structure Model

In this section we formulate the Financial market model used to describe the movement of defaultable and non-defaultable bond prices over time. To do that we fix a terminal time horizon T^*. Uncertainty in the financial market is modelled by a complete probability space $(\Omega, \mathcal{F}, \mathbf{Q})$ and a filtration $\mathbf{F} = (\mathcal{F}_t)_{0 \le t \le T^*}$ of the three-dimensional standard Brownian motion $\hat{W}(t)' = \left(\hat{W}_r(t), \hat{W}_s(t), \hat{W}_u(t)\right)$ satisfying the usual conditions, i.e. $\mathcal{F}_0$ is trivial and contains all the $\mathbf{Q}$–null sets of $\mathcal{F}$ and the filtration $\mathbf{F}$ is right continuous. In addition, we assume $\mathcal{F}_{T^*} = \mathcal{F}$. The filtration $\mathbf{F}$ represents the arrival of information over time. As numéraire we choose the money market account $B(t)$ with $\ln B(t) = \int_0^t r(\tau)\, d\tau$, where $r(t)$ is the non-defaultable short rate

at time $t \in [0, T^*]$. The modelling takes already place after measure transformation, i.e. we assume that $\mathbf{Q}$ is a martingale measure and all discounted security price processes are local $\mathbf{Q}$–martingales with respect to our numéraire.

Assumption 1 *The dynamics of the non-defaultable short rate is given by the following stochastic differential equation:*

$$dr(t) = [\theta_r(t) - \hat{a}_r \cdot r(t)]\, dt + \sigma_r\, d\hat{W}_r(t), \ t \in [0, T^*], \tag{1}$$

where $\hat{a}_r$, $\sigma_r > 0$ *are positive constants and* θ_r *is a non-negative valued deterministic function.*

This specification implies that the current rate $r(t)$ is pulled towards $\frac{\theta_r(t)}{\hat{a}_r}$ with a speed of adjustment $\hat{a}_r$.

Assumption 2 *The development of the uncertainty index is given by the following stochastic differential equation:*

$$du(t) = [\theta_u - \hat{a}_u \cdot u(t)]\, dt + \sigma_u \cdot \sqrt{u(t)} d\hat{W}_u(t), \ t \in [0, T^*], \tag{2}$$

where $\hat{a}_u$, $\sigma_u > 0$ *are positive constants and* θ_u *is a non-negative constant.*

The short rate spread is defined to be the defaultable short rate minus the non-defaultable short rate and is supposed to hold the following assumption.

Assumption 3 *The dynamics of the short rate spread is given by the following stochastic differential equation:*

$$ds(t) = [b_s \cdot u(t) - \hat{a}_s \cdot s(t)]\, dt + \sigma_s \cdot \sqrt{s(t)} d\hat{W}_s(t), \ t \in [0, T^*], \tag{3}$$

where $\hat{a}_s$, $b_s, \sigma_s > 0$ *are positive constants.*

Given assumption 1, the price of a non-defaultable zero coupon or discount bond is given by the following Theorem (see (Hull & White 1990)).

Theorem 1 *Under assumption 1 the time t value P (t, T) = P (r, t, T) of a non-defaultable discount bond with maturity T ∈ [t, T*], is given by*

$$P(t,T) = A(t,T)\, e^{-B(t,T)\cdot r(t)},\ t \in [0,T], \tag{4}$$

where A (t, T) and *B (t, T)* are defined by

$$B(t,T) = \frac{1}{\hat{a}_r} \cdot \left[1 - e^{-\hat{a}_r \cdot (T-t)} \right] \tag{5}$$

and

$$\begin{aligned} \ln A(t,T) &= \int_t^T \left(\frac{1}{2}\sigma_r^2 \cdot B^2(\tau,T) - \theta_r(\tau) \cdot B(\tau,T) \right) d\tau \\ &= \ln\left(\frac{P(0,T)}{P(0,t)} \right) - B(t,T) \cdot \frac{\partial \ln P(0,t)}{\partial t} \\ &\quad - \frac{\sigma_r^2}{4\hat{a}_r^3} \cdot \left(e^{-\hat{a}_r T} - e^{-\hat{a}_r t}\right)^2 \cdot \left(e^{2\hat{a}_r t} - 1\right). \end{aligned} \tag{6}$$

The deterministic function $\theta_r(t)$ is given by

$$\theta_r(t) = f_t(0,t) + \hat{a}_r \cdot f(0,t) + \frac{\sigma_r^2}{2\hat{a}_r} \cdot \left(1 - e^{-2\hat{a}_r t}\right),$$

where *f (0, t)* denotes the instantaneous forward rate at time *t* as seen from time 0. The (continuous) yield a *R (t, T)* of a non-defaultable discount bond, also known as non-defaultable (continuous) zero rate, at time *t* ∈ *[0, T]* for a maturity time *t* ∈ [*t, T**] is given by

$$R(t,T) = -\frac{1}{T-t} \cdot \ln P(t,T) = a(t,T) + b(T-t) \cdot r(t) \tag{7}$$

with

$$a(t,T) = -\frac{\ln A(t,T)}{T-t} \quad \text{and} \quad b(T-t) = \frac{B(t,T)}{T-t}.$$

Under the additional assumption that the three driving Wiener processes are independent, Schmid & Zagst (2000) generalized the result of Theorem 1 to the

pricing of defaultable zero coupon bonds. To show their result, let $T^d \in [0, \infty)$ denote the **F**–stopping time describing the stochastic behaviour of the default time and let us assume a fractional recovery of market value in the case of default. Then the following Theorem holds:

Theorem 2 *Assuming the dynamics specified by equations (1), (2) and (3), the value $P^d(t, T) = P^d(r, s, u, t, T)$ of a defaultable discount bond at time t < T =* min *(T, T^d), T ∈* [*t, T**] is given by

$$P^d(t,T) = A^d(t,T)\, e^{-B(t,T)\cdot r(t) - C^d(t,T)\cdot s(t) - D^d(t,T)\cdot u(t)} \tag{8}$$

where

$$\begin{aligned} C^d(t,T) &= \frac{1 - e^{-\delta_s \cdot (T-t)}}{\kappa_1^{(s)} - \kappa_2^{(s)} e^{-\delta_s \cdot (T-t)}}, \\ D^d(t,T) &= \frac{-2v'(t,T)}{\sigma_u^2 \cdot v(t,T)}, \\ \ln A^d(t,T) &= \ln A(t,T) + \frac{2\theta_u}{\sigma_u^2} \cdot \ln \left| \frac{v(T,T)}{v(t,T)} \right|, \end{aligned}$$

and v (t, T) is defined by

$$\begin{aligned} v(t,T) = \left(\sigma_u^2\right)^{\frac{\hat{a}_u}{2\delta_s} + \phi\left(\kappa_1^{(s)}\right)} \cdot \quad & \left[\frac{\alpha_1}{\alpha_2} \cdot \left(e^{-\delta_s \cdot (T-t)}\right)^{\frac{\hat{a}_u}{2\delta_s} - \phi\left(\kappa_1^{(s)}\right)} \cdot F_1(t,T) \right. \\ & \left. + \left(e^{-\delta_s \cdot (T-t)}\right)^{\frac{\hat{a}_u}{2\delta_s} + \phi\left(\kappa_1^{(s)}\right)} \cdot F_3(t,T) \right], \end{aligned}$$

with

$$
\begin{aligned}
F_1(t,T) &= F\left(-\phi\left(\kappa_1^{(s)}\right)-\phi\left(\kappa_2^{(s)}\right), -\phi\left(\kappa_1^{(s)}\right)+\phi\left(\kappa_2^{(s)}\right),\right.\\
&\qquad \left.1-2\phi\left(\kappa_1^{(s)}\right), \kappa_2^{(s)}/\kappa_1^{(s)}e^{-\delta_s\cdot(T-t)}\right),\\
F_3(t,T) &= F\left(\phi\left(\kappa_1^{(s)}\right)-\phi\left(\kappa_2^{(s)}\right), \phi\left(\kappa_1^{(s)}\right)+\phi\left(\kappa_2^{(s)}\right),\right.\\
&\qquad \left.1+2\phi\left(\kappa_1^{(s)}\right), \kappa_2^{(s)}/\kappa_1^{(s)}e^{-\delta_s\cdot(T-t)}\right),
\end{aligned}
$$

and

$$
\delta_s=\sqrt{\hat{a}_s^2+2\sigma_s^2},\ \kappa_{1/2}^{(s)}=\frac{\hat{a}_s\pm\delta_s}{2},\ \phi(g)=\sqrt{\frac{\hat{a}_u^2 g+2b_s\sigma_u^2}{4\delta_s^2 g}}.
$$

In addition, $\alpha_1=\alpha_1(T,T)$, *and,* $\alpha_2=\alpha_2(T,T)$, *where*

$$
\begin{aligned}
\alpha_1(t,T) &= \zeta_2\cdot e^{-\delta_s\cdot(T-t)}\cdot F_4(t,T)-\xi_1\cdot F_3(t,T),\\
\alpha_2(t,T) &= \xi_2\cdot F_1(t,T)-\zeta_1\cdot e^{-\delta_s\cdot(T-t)}\cdot F_2(t,T),
\end{aligned}
$$

with

$$
\xi_{1/2}=\left(\frac{\hat{a}_u}{2}\pm\delta_s\cdot\phi\left(\kappa_1^{(s)}\right)\right),\ \zeta_{1/2}=\delta_s\cdot\frac{\kappa_2^{(s)}}{\kappa_1^{(s)}}\cdot\frac{\phi^2\left(\kappa_2^{(s)}\right)-\phi^2\left(\kappa_1^{(s)}\right)}{1\mp 2\phi\left(\kappa_1^{(s)}\right)},
$$

and

$$
\begin{aligned}
F_2(t,T) &= F\left(1-\phi\left(\kappa_1^{(s)}\right)-\phi\left(\kappa_2^{(s)}\right), 1-\phi\left(\kappa_1^{(s)}\right)+\phi\left(\kappa_2^{(s)}\right),\right.\\
&\qquad \left.2-2\phi\left(\kappa_1^{(s)}\right), \kappa_2^{(s)}/\kappa_1^{(s)}e^{-\delta_s\cdot(T-t)}\right),\\
F_4(t,T) &= F\left(1+\phi\left(\kappa_1^{(s)}\right)-\phi\left(\kappa_2^{(s)}\right), 1+\phi\left(\kappa_1^{(s)}\right)+\phi\left(\kappa_2^{(s)}\right),\right.\\
&\qquad \left.2+2\phi\left(\kappa_1^{(s)}\right), \kappa_2^{(s)}/\kappa_1^{(s)}e^{-\delta_s\cdot(T-t)}\right).
\end{aligned}
$$

Hereby, $F(a,b,c,z)$ *denotes the hypergeometric function, i.e.*

$$
F(a,b,c,z)=\frac{\Gamma(c)}{\Gamma(b)\cdot\Gamma(c-b)}\int_0^1 t^{b-1}\cdot(1-t)^{c-b-1}\cdot(1-t\cdot z)^{-a}\,dt,\ c>b>0.
$$

The (continuous) yield $R^d(t,T)$ of a defaultable discount bond, also known as defaultable (continuous) zero rate, at time $t\in[0,T]$ for a maturity time $T\in[t,T^*]$ is given by

$$
\begin{aligned}
R^d(t,T) &= -\frac{1}{T-t}\cdot\ln P^d(t,T)\\
&= -\frac{\ln A(t,T)}{T-t}-\frac{2\theta_u}{\sigma_u^2\cdot(T-t)}\cdot\ln\left|\frac{v(T,T)}{v(t,T)}\right|+\frac{B(t,T)}{T-t}\cdot r(t)\\
&\quad +\frac{C^d(t,T)}{T-t}\cdot s(t)+\frac{D^d(t,T)}{T-t}\cdot u(t).
\end{aligned}
$$

Hence, the credit spread $S(t,T)$ at time $t\in[0,T]$ for a maturity time $T\in[t,T^*]$ can be calculated as

$$
\begin{aligned}
S(t,T) &= R^d(t,T)-R(t,T)\\
&= a^d(T-t)+c(T-t)\cdot s(t)+d(T-t)\cdot u(t) \qquad (9)
\end{aligned}
$$

with

$$
a^d(T-t)=-\frac{2\theta_u}{\sigma_u^2}\cdot\frac{\ln\left|\frac{v(T,T)}{v(t,T)}\right|}{T-t},\ c(T-t)=\frac{C^d(t,T)}{T-t},
$$

and

$$d(T-t) = \frac{D^d(t,T)}{T-t}.$$

The non-defaultable zero rates and the credit spreads for different maturities will be derived from market data as described in the following section.

3 Data Set and Parameter Estimation

In the following analysis we use observed data of credit spreads between AAA-rated German and AA-rated Italian as well as AAA-rated German and (A-)-rated Greek government bonds. We assume throughout that the German bonds are default risk free whereas the Italian and Greek bonds are defaultable. We consider a 12 months time series of daily bond prices from November 1, 1999, until October 23, 2000, provided by Reuters Information Services[2]. All prices are denominated in Euros, so we don't have to take care of possible currency risks involved in the credit spreads. Note further that our sample for Germany and Italy only consists of pure discount bonds. The Greek bonds are all fixed rate bonds paying annual coupons between 3% p.a. and 9.7% p.a. We use a universe of 33 German, 36 Italian, and 38 Greek bonds. The ranges of maturities of the bonds span from 0.36 years to 20.8 years. We apply the four parameter approximation of Nelson & Siegel (1987) to infer the whole zero rate curves from the bond data (for details see (Schmid & Kalemanova 2002)).

In the following we use Kalman filtering methods to estimate the parameters of the three-factor defaultable term structure model from the zero curves. For details on the methodology see, e.g., (Schmid & Kalemanova 2002). The application of Kalman filtering methods to the parameter estimation of term structure models has the great advantage that it allows the underlying state variables to be handled as completely unobservable, whereas other methods often must use some short term rates as proxies and hereby introduce additional noise. Because we use real world credit spread observations in the Kalman filter we must describe the credit spread dynamics under the "real world" measure which we denote by $\mathbf{P}$. Our assumptions for the short rate as well as the uncertainty index and short rate credit spread for Italy and Greece are summarized as follows: Under the measure $\mathbf{P}$ the processes r, s^j, and u^j, $j \in \{I, G\}$, are given by

$$\begin{aligned} dr(t) &= [\theta_r(t) - a_r r(t)]\,dt + \sigma_r dW_r(t), \\ du^j(t) &= \left[\theta_u^j - a_u^j u^j(t)\right] dt + \sigma_u^j \sqrt{u^j(t)}\,dW_u^j(t), \\ ds^j(t) &= \left[b_s^j u^j(t) - a_s^j s^j(t)\right] dt + \sigma_s^j \sqrt{s^j(t)}\,dW_s^j(t), \quad 0 \le t \le T^*, \end{aligned} \tag{10}$$

where

$$\hat{a}_i^j = a_i^j + \lambda_i^j \cdot \left(\sigma_i^j\right)^2, \hat{W}_r(t) = W_r(t) + \int_0^t \gamma_r(\tau)\,d\tau,$$

and

$$\hat{W}_i^j(t) = W_i^j(t) + \int_0^t \gamma_i^j(\tau)\,d\tau, i = s, u$$

with

$$\begin{aligned} \gamma_r(t) &= \lambda_r \sigma_r r(t), \\ \gamma_s^j(t) &= \lambda_s^j \sigma_s^j \sqrt{s^j(t)}, \text{ and} \\ \gamma_u^j(t) &= \lambda_u^j \sigma_u^j \sqrt{u^j(t)}, \ 0 \le t \le T^*. \end{aligned}$$

The measurement equation for the Kalman filter is set up using equation (7) for different maturities and the short rate r as state variable for an estimation of the parameters θ_r, a_r, σ_r, and λ_r. For estimating the parameters $\theta_u^j, a_u^j, \sigma_u^j, \lambda_u^j, b_s^j, a_s^j, \sigma_s^j$, and λ_s^j we use equation (9) for different maturities with $u = u^j$ and $s = s^j$ as well as (u^j, s^j) as state variable, $j \in \{I, G\}$.

2 Most of the previous empirical studies use weekly or monthly data. Besides Duellmann and Windfuhr (Duellmann & Windfuhr 2000) we are the only ones to use daily data – at least up to our knowledge.

For more details see (Schmid & Kalemanova 2002). For Germany the results of the Kalman filter estimations are the following: $\theta_r = 0.014413$, $a_r = 0.238205$, $\sigma_r = 0.015581$, $\lambda_r = -0.086076$, $r(0) = 0.042434$. For Italy: $b_s^I = 0.274800$, $a_s^I = 0.047687$, $\sigma_s^I = 0.158324$, $\lambda_s^I = -1.898668$, $s^I(0) = 0.001296$, $\theta_u^I = 0.000031$, $a_u^I = 0.068696$, $\sigma_u^I = 0.030482$, $\lambda_u^I = -1.228143$, $u^I(0) = 0.005112$. For Greece: $b_s^G = 0.343225$, $a_s^G = 0.167814$, $\sigma_s^G = 0.446885$, $\lambda_s^G = -0.215937$, $s^G(0) = 0.002325$, $\theta_u^G = 0.000392$, $a_u^G = 0.074952$, $\sigma_u^G = 0.067231$, $\lambda_u^G = -1.109091$, $u^G(0) = 0.012704$.

4 Portfolio Optimization

In this section we deal with the problem of maximizing the final value of a portfolio under limited risk. We hereby assume that we are given a planning horizon $T \in [0, T^*]$ and that the prices at any time $t \in [0, T]$ of the n sovereign bonds we consider are given by $P_1(t), \ldots, P_n(t)$. The first source of risk is the changing interest rate over time resulting in varying values of the portfolio $\varphi = (\varphi_1, \ldots, \varphi_n)$. The second source of risk is the possibility of default inherent in the sovereign bonds of the countries $j \in \{I, G\}$. Due to this risk we may lack in coupon payments which we do not receive in case of default. The first limit we set is a restriction on the portfolio value falling below some given benchmark at times $t \in \mathcal{T}_B = \{T_1^B, \ldots, T_{m_B}^B\} \subset [0, T]$, $m_B \in \mathbb{N}$. The second limit is due to a stream of liabilities $L(t)$ at times $t \in \mathcal{T}_L = \{T_1^L, \ldots, T_{m_L}^L\} \subset [0, T]$, $m_L \in \mathbb{N}$, which have to be covered by the coupon payments of our bond portfolio. Because these coupon payments are under default risk we also set a restriction on the cash flows falling below the corresponding liability. Coupon payments we receive between the liability payment dates are put in a cash account at a continuous interest of R_{ca}, i.e. a cash flow $C_i(t)$ received from bond $i \in \{1, \ldots, n\}$ at time $t \in (T_{\Upsilon-1}^L, T_\Upsilon^L]$, will have a cash value of

$$C_i\left(t, T_\Upsilon^L\right) = C_i(t) \cdot e^{R_{ca}\left(t, T_\Upsilon^L\right) \cdot \left(T_\Upsilon^L - t\right)}$$

at time T_Υ^L on the cash account. The value of the cash account at time $t \in [0, T]$ is denoted by $C_0(t)$. We suppose that country $j \in \{I, G\}$ enters default as soon as the corresponding uncertainty process u^j crosses the default boundary $\xi^{d,j}$ evaluated from the default probability of country j. Therefore, the time of the default event is given by

$$T^{d,j} = \inf\left\{t \in [0, T] : u^j(t) > \xi^{d,j}\right\}.$$

In the case of default before time T the holder of the "defaulted" coupon bond $P_i(t) = P_c^{d,j}(t, T_i)$ with maturity $T_i \in [T^{d,j}, T^*]$ is assumed to receive no more coupon payments but a recovery payment of $w \cdot P_c^{d,j}(T^{d,j}-, T_i)$ fixed at time $T^{d,j}$ with $w \in [0, 1]$ denoting the recovery rate and $P_c^{d,j}(T^{d,j}-, T_i)$ being the price of the bond just before default. For simplicity we suppose that this payment is available not before the maturity time T_i of the bond at a cash value of

$$Z\left(T^{d,j}, T_i\right) = e^{R_{ca}^{d,j}\left(T^{d,j}, T_i\right) \cdot \left(T_i - T^{d,j}\right)} \cdot w \cdot P_c^{d,j}\left(T^{d,j}-, T_i\right)$$

with $R_{ca}^{d,j}(T^{d,j}, T_i)$ denoting the interest rate earned on the recovery payment for the time period $[T^{d,j}, T_i]$. Hence, at each of the coupon payment dates $0 \leq t_{i1} < \ldots < t_{in_i} = T_i$, the cash flows of the defaultable bond are given by

$$C_i^d(t_{i\nu}) = C_i(t_{i\nu}) \cdot \left(1 - 1_{[0, t_{i\nu}]}\left(T^{d,j}\right)\right)$$

if $\nu \in \{1, \ldots, n_i - 1\}$ and

$$C_i^d(T_i) = C_i(T_i) \cdot \left(1 - 1_{[0, T_i]}\left(T^{d,j}\right)\right) + 1_{[0, T_i]}\left(T^{d,j}\right) \cdot Z\left(T^{d,j}, T_i\right),$$

where $1_{[0,t]}$ represents the usual indicator variable. The cash flows within each liability period $(T^L_{\Upsilon-1}, T^L_\Upsilon]$, $\Upsilon \in \{1, ..., m_L\}$, are then put in the cash account accruing to a cash value of

$$C_{i0}\left(T^L_{\Upsilon-1}, T^L_\Upsilon\right) = \sum_{\substack{\nu \in \{1,...,n_i\} \\ t_{i\nu} \in (T^L_{\Upsilon-1}, T^L_\Upsilon]}} C^d_i\left(t_{i\nu}\right) \cdot e^{R_{ca}\left(t_{i\nu}, T^L_\Upsilon\right) \cdot \left(T^L_\Upsilon - t_{i\nu}\right)}$$

at the liability payment dates $T^L_\Upsilon \in \mathcal{T}_L$. Hence, the cash value at time $T^L_\Upsilon \in \mathcal{T}_L$ of the portfolio cash flows within period $(T^L_{\Upsilon-1}, T^L_\Upsilon]$, $\Upsilon \in \{1, ..., m_L\}$, is given by

$$C_0\left(\varphi, T^L_{\Upsilon-1}, T^L_\Upsilon\right) = \sum_{i=1}^{n} \varphi_i \cdot C_{i0}\left(T^L_{\Upsilon-1}, T^L_\Upsilon\right)$$

leading to a total value of the cash account at time T^L_Υ of

$$C_0\left(\varphi, T^L_\Upsilon\right) = C_0\left(\varphi, T^L_{\Upsilon-1}\right) \cdot e^{R_{ca}\left(T^L_{\Upsilon-1}, T^L_\Upsilon\right) \cdot \left(T^L_\Upsilon - T^L_{\Upsilon-1}\right)} + C_0\left(\varphi, T^L_{\Upsilon-1}, T^L_\Upsilon\right) - L\left(T^L_\Upsilon\right)$$

for all $\Upsilon \in \{1, ..., m_L\}$ where we set

$$C_0\left(\varphi, 0\right) = \varphi_0\left(0\right) = C_0 - \sum_{i=1}^{n} \varphi_i \cdot P_i\left(0\right)$$

with C_0 denoting the initial budget to be invested at time $t = 0$.

Having estimated the parameters of the stochastic processes r, s^I, u^I, s^G, and u^G we now consider the vector of risk factors

$$\mathsf{RF} = (\mathsf{RF}_1, \ldots, \mathsf{RF}_5) = (r, s^I, u^I, s^G, u^G).$$

The optimization problem is based on a simulation of RF over time. For this simulation, let $V(0)$ and $V(t)$ denote the value of a given stochastic variable V at times 0 and t. In our case this may be the cash account or the dirty price of a

bond plus potential coupon payments between times 0 and t, at time 0 and time t as seen from time 0. According to our model we know that the values $V(t)$ are dependent on the vector RF of risk factors and that the functional relation between the risk vector and the future value of V is known, i.e. $V(t) = V(\mathsf{RF}, t)$. Under this assumption we simulate the risk vector RF getting the simulations

$$\mathsf{RF}^k = \left(\mathsf{RF}^k_1, \ldots, \mathsf{RF}^k_5\right) \text{ with probability } p_k > 0,\ k = 1, \ldots, K.$$

Inserting these simulations we receive the simulations for the future values

$$V^k(t) = V\left(\mathsf{RF}^k, t\right),\ k = 1, \ldots, K.$$

For any portfolio $\varphi = (\varphi_1, ..., \varphi_n)$ of the bonds, which we consider to be fixed from 0 to the end of the planning horizon T for the ease of exposition, the future value of the portfolio at time t is denoted by $V(\varphi, t)$. If $V(\varphi, t)$ models the cash account of the portfolio we have $V(\varphi, t) = C_0(\varphi, t)$. If $V_i(t)$ models the dirty price of bond $i \in \{1, \ldots, n\}$ plus potential coupon payments between times 0 and t, at time t as seen from time 0, we have

$$V(\varphi, t) = \sum_{i=1}^{n} \varphi_i \cdot V_i(t)$$

with $V(\varphi, 0)$ denoting the cash account or portfolio value at time 0. Using our simulations, the future value $V(\varphi, t)$ is simulated by

$$V^k(\varphi, t) = C^k_0(\varphi, t),\ k = 1, ..., K,$$

in case of the cash account and

$$V^k(\varphi, t) = \sum_{i=1}^{n} \varphi_i \cdot V^k_i(t),\ k = 1, ..., K,$$

in case of the coupon payments. To restrict the downside risk of the future value at time $t \in \mathcal{T}_B$ we consider the discrete version of the lower partial moment of order $l \in I\!N$ corresponding to an investor specific benchmark[3] $B(t) \in I\!R$ which is defined by

$$LPM_l(\varphi, V, B, t) = \sum_{\substack{k=1,\dots,K \\ V^k(\varphi,t)<B(t)}} p_k \cdot \left(B(t) - V^k(\varphi, t)\right)^l. \qquad (11)$$

The lower partial moment only considers realizations of the future value of V below the investor specific benchmark measured to a power of l. For $l = 0$ this is the probability that the random future value falls below the given benchmark which is referred to as shortfall probability. Setting the benchmark equal to $V(\varphi, 0)$ this is the probability of loss. For $l = 1$, the lower partial moment is the expected deviation of the future values below the benchmark, sometimes called (expected) regret. For $l = 2$, the lower partial moment is weighting the squared deviations below the benchmark and thus is the semi-variance if the benchmark is set equal to the expected future value. For a more detailed discussion of the lower partial moments see, e.g., (Harlow 1991) or (Zagst 2002).

A portfolio manager or trader may be restricted to specific trading limits. Therefore, we introduce the absolute lower (s_i) and upper bounds (S_i) for the amount φ_i of security $i = 1, \dots, n$ in the portfolio and claim that

$$s_i \le \varphi_i \le S_i, \; i = 1, \dots, n.$$

3 If we are given a benchmark return $b(t)$ for the period $[0, t]$, the corresponding absolute benchmark $B(t)$ to be compared with the simulated portfolio values $V^k(\varphi, t)$ is given by $B(t) = V(\varphi, 0) \cdot (1 + b(t))$.

For $A_{Bl}(t), A_{Ll}(t) \in I\!R$ and $l \in \{0, 1, 2,\}$, let us consider the following optimization problem

$$(P)\begin{cases} \sum_{i=1}^{n} \varphi_i \cdot E_Q[V_i(T)] \to \max \\ LPM_l(\varphi, V, B, t) \le A_{Bl}(t), \; l \in \{0,1,2\}, t \in T_B \\ LPM_l(\varphi, C_0, 0, t) \le A_{Ll}(t), \; l \in \{0,1,2\}, t \in T_L \\ s_i \le \varphi_i \le S_i, \; i = 0, \dots, n \\ \varphi_0 = C_0 - \sum_{i=1}^{n} \varphi_i \cdot V_i(0) \end{cases}$$

with $V(t)$, $t \in [0, T]$, in this case, denoting the dirty price of the portfolio plus potential coupon payments between times 0 and t, at time t as seen from time 0. Note, that we have set $R_{ca}\left(T^L_{\Upsilon-1}, T^L_{\Upsilon}\right) \equiv R_{ca}$ for the ease of exposition. To implement this problem let us have a closer look at the general LPM constraint, i.e. the LPM restriction for the future value V, first. We choose the numbers $m_k < 0$ to be sufficiently small and the numbers $M_k > 0$, $m_{kl} > 0$ and $M_{kl} > 0$ to be sufficiently large and define the constraints

$$M_k \cdot y_k + V^k(\varphi, t) \ge B(t) \qquad (A)$$

$$m_k \cdot (1 - y_k) + V^k(\varphi, t) < B(t) \qquad (B)$$

$$0 \le \left(V^k(\varphi, t) - B(t)\right)^l + (-1)^{l-1} \cdot w_{kl} \le M_{kl} \cdot (1 - y_k) \qquad (C)$$

$$0 \le w_{kl} \le m_{kl} \cdot y_k \qquad (D)$$

with $w_{kl} \in I\!R$, $y_k \in \{0, 1\}$ for all $k = 1, \dots, K$, where we consider $l \in \{0, 1, 2,\}$ to be arbitrary but fixed as well as $t \in (0, T]$. A proof of the following Theorem can be found in (Zagst 2002).

Theorem 3 *Let $t \in (0, T]$, $l \in \{0, 1, 2\}$ be arbitrary but fixed and $y_k \in \{0, 1\}$ for all $k = 1, \ldots, K$.*

a) Let condition (A) be satisfied. Then we have for all $k = 1, \ldots, K$:

$$y_k = 1 \quad \text{if} \quad V^k(\varphi, t_0, T) < B(t_0, T).$$

Let condition (B) be satisfied. Then we have for all $k = 1, \ldots, K$:

$$y_k = 0 \quad \text{if} \quad V^k(\varphi, t_0, T) \geq B(t_0, T).$$

b) Let conditions (A) and (B) be satisfied. Then, for all $k = 1, \ldots, K$ we have:

$$V^k(\varphi, t_0, T) < B(t_0, T) \quad \text{if and only if } y_k = 1.$$

c) Under conditions (A), (C), and (D), we have[4]*:*

$$LPM_l(\varphi, V, B, t) \leq \sum_{k=1}^{K} p_k \cdot w_{kl}.$$

d) Under conditions (A), (B), (C), and (D), we have:

$$LPM_l(\varphi, V, B, t_0, T) = \sum_{k=1}^{K} p_k \cdot w_{kl}.$$

Note that for the special case $l = 0$, we can conclude that conditions (*C*) and (*D*) are equivalent to $w_{k0} = y_k$ giving us

$$LPM_0(\varphi, V, B, t) = \sum_{k=1}^{K} p_k \cdot y_k \tag{12}$$

4 To be precise, we only need the first inequality of (*D*) in addition to (*A*) and (*C*) to get this statement. Furthermore, we only need the first inequalities of (*C*) and (*D*) if $l = 1$.

under the additional conditions (*A*) and (*B*), with "≤" instead of "=" if only condition (*A*) is satisfied in addition to (*C*) and (*D*). Using Theorem 3 we can replace the lower partial moment constraint

$$LPM_l(\varphi, V, B, t) \leq A_l \tag{LPM}$$

by the constraint

$$\sum_{k=1}^{K} p_k \cdot w_{kl} \leq A_l \tag{E}$$

for $A_l \in \mathbb{R}$ and $l \in \{0, 1, 2\}$, if conditions (*A*), (*B*), (*C*), and (*D*) are satisfied. Hence, instead of using constraint (LPM), we can use inequality (*E*) if we add conditions (*A*), (*B*), (*C*), and (*D*) to the optimization problem. Furthermore, we can omit condition (*B*) for a sufficient set of conditions. For $l = 0$ condition (*LPM*) is called shortfall constraint and the corresponding A_0, in this case chosen to be an element of (0, 1), is called shortfall probability.

Usually all commercial optimization tools use inequalities of the form ≤ or and ≥ a precision expressed by the smallest absolute number that can be recognized numerically within the tool and which will be denoted by $\varepsilon > 0$ here. For this reason we rewrite equation (*B*) in the following form

$$m_k \cdot (1 - y_k) + V^k(\varphi, t) \leq B(t) - \varepsilon. \tag{B'}$$

Let us therefore denote for $l \in \{0, 1, 2\}$ and $t \in (0, T]$ the set of restrictions (*A*), (*B'*), (*C*), and (*D*) by

$$MIP_l(\varphi, y_k, w_{kl}, V^k, B, t), \; k = 1, \ldots, K.$$

This gives us the following optimization problem (P_1) which is (approximately) equivalent to (P):

$$
(P_1)\begin{cases}
\sum_{k=1}^{K} p_k \cdot V^k(\varphi, T) \to \max \\
MIP_l(\varphi, y_k, w_{kl}, V^k, B, t), \ k = 1, ..., K, t \in T_B \\
MIP_l(\varphi, y_{0k}, w_{0kl}, C_0^k, 0, t), \ k = 1, ..., K, t \in T_L \\
y_k, y_{0k} \in \{0,1\}, \ k = 1, ..., K \\
w_{kl}, w_{0kl} \in I\!R \\
\sum_{k=1}^{K} p_k \cdot w_{kl} \le A_{Bl}, \ l \in \{0,1,2\} \\
\sum_{k=1}^{K} p_k \cdot w_{0kl} \le A_{Ll}, \ l \in \{0,1,2\} \\
s_i \le \varphi_i \le S_i, \ i = 1, ..., n \\
\varphi_0 = C_0 - \sum_{i=1}^{n} \varphi_i \cdot V_i(0).
\end{cases}
$$

The variables of (P_1) to be optimized are $\varphi_0, ..., \varphi_n$, $y_k, y_{0k} \in \{0,1\}$, and $w_{kl}, w_{0kl} \in I\!R$, $k = 1, ..., K$. Hence, we have to solve a linear mixed-integer program which can be done by commercial optimization tools.

Remark. Using the simulated values $u^{k,j}$, $k = 1, \ldots, K$, we can calculate the simulated probabilities of default at time $t \in [0, T]$ by

$$p^{d,j}(t) = \sum_{k=1}^{K} p_k \cdot 1_{[0,t]}\left(T^{d,j}\right)$$

with

$$T^{d,j} := \inf\left\{t \in [0,T] : u^{k,j}(t) > \xi^{d,j}\right\}, \ k = 1, ..., K.$$

Let us finally mention that there are optimization models related to ours. See, e.g., (Dembo 1993) and (Mausser & Rosen 1999). Dembo does not explicitly consider defaults, but describes scenario-based optimization models for bond portfolios, including cash flow matching. Mausser and Rosen construct e.cient frontiers for a portfolio of bonds under credit risk using various scenario-based optimization models but they do not consider cash flow tracking.

i	Country	Maturity	Coupon	Dirty Price
1	Germany	1.0Y	6%	101.40
2	Germany	1.5Y	6%	104.92
3	Germany	2.0Y	6%	102.42
4	Germany	2.5Y	6%	104.92
5	Germany	3.0Y	6%	103.17
6	Italy	1.0Y	7%	100.94
7	Italy	1.5Y	7%	104.88
8	Italy	2.0Y	7%	101.91
9	Italy	2.5Y	7%	104.80
10	Italy	3.0Y	7%	102.79
11	Greece	1.0Y	8%	101.96
12	Greece	1.5Y	8%	106.74
13	Greece	2.0Y	8%	103.54
14	Greece	2.5Y	8%	107.02
15	Greece	3.0Y	8%	104.79

Table 1: Specification of the portfolio universe

5 Case Study

For the case study we have chosen a planning horizon of $T = 2$ years and a number of 5 sovereign bonds with annual coupon payments of each of the countries Germany, Italy, and Greece, i.e. $n = 15$. The specification and dirty prices at time $t = 0$ of the bonds with a notional of 100 *Euro* each are summarized in table 1.

Furthermore, we set $\mathcal{T}_B = \mathcal{T}_L = \{6M, 1Y, 1Y6M, 2Y\}$ with a liability stream of (50, 000; 100, 000; 200, 000; 400, 000) *Euro*, $R_{ca} = R_{ca}^{d,I} = R_{ca}^{d,G} = 0$, and a budget of $C_0 = 1$ *Mio. Euro*. As mentioned above, $V_i(t)$ is considered to be the dirty price of bond $i \in \{1, \ldots, n\}$ plus accumulated cash flows over time accrued at an interest of R_{ca}. For the optimization we have chosen $K = 100$ simulations, absolute lower bounds of $s_i = 0$ and upper bounds of $S_i = +\infty$, $i \in \{1, \ldots, n\}$, a benchmark of $B(t) = C_0$ for all $t \in \mathcal{T}_L$, $l = 0$, i.e. we are dealing with shortfall constraints, and $A_B^0(t) = 1\% = A_L^0(t)$. The probabilities of default and the resulting default boundaries $\xi^{d,I}$ and $\xi^{d,G}$ are given in table 2.

According to the rather low default probabilities, no default took place in our simulations. Therefore, the shortfall constraint for covering the liability stream degenerates to an exact cash flow matching restriction as stated, e.g., in (Elton & Gruber 1991), p. 565–566.

	Default probability	Default boundary
Italy	0.02 %	0.014659
Greece	0.04 %	0.047047

Table 2: Default probabilities and boundaries for Italy and Greece

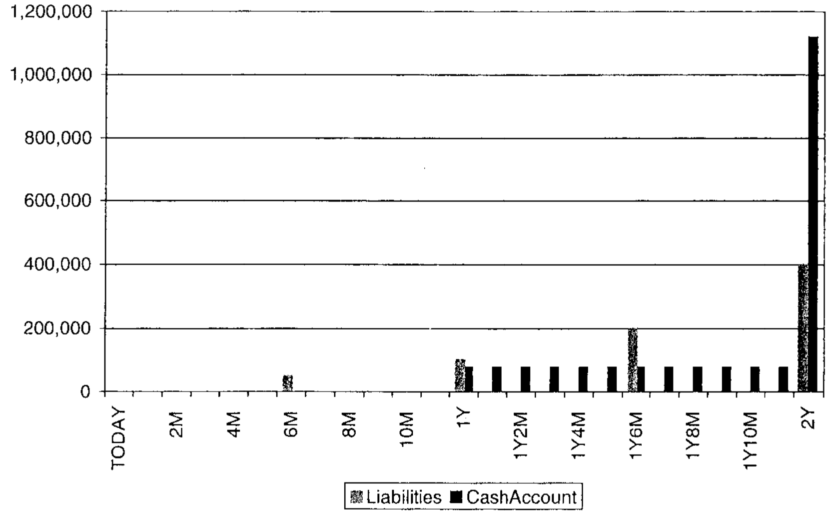

Figure 1: Value of the cash account in example 1 over time compared to the liability stream

Example 1. In the first example we do not consider any shortfall or liability constraints. Our aim is simply to maximize the expected final portfolio value, i.e. the dirty price of the portfolio plus the value of the cash account at time T, allowing for a budget of C_0. The optimal portfolio consists of an amount of 9,658, i.e. a cash value of 1 *Mio. Euro,* of the 2 year 8 % Greek bond. This portfolio has an expected final value (including previous coupon payments) of 1,120,337 *Euro* corresponding to an expected rate of return of 5.68 %. The probabilities of falling below the benchmark C_0 at times $t \in \mathcal{T}_L$ are given by the vector (5 %, 2 %, 1 %, 0 %). Also, the liability stream is not covered by this portfolio. The value of the cash account compared to the liability stream is shown in figure 1.

Example 2. For the second example we add the shortfall constraints for the portfolio value at times $t \in \mathcal{T}_L$ to the budget constraint of example 1 getting an optimal portfolio which consists of

3,008 (311,428 *Euro*) of the 2 year 8 % Greek bond
2,485 (253,287 *Euro*) of the 2 year 7 % Italian bond
3,604 (369,108 *Euro*) of the 2 year 6 % German bond
641 (66,177 *Euro*) of the 3 year 6 % German bond.

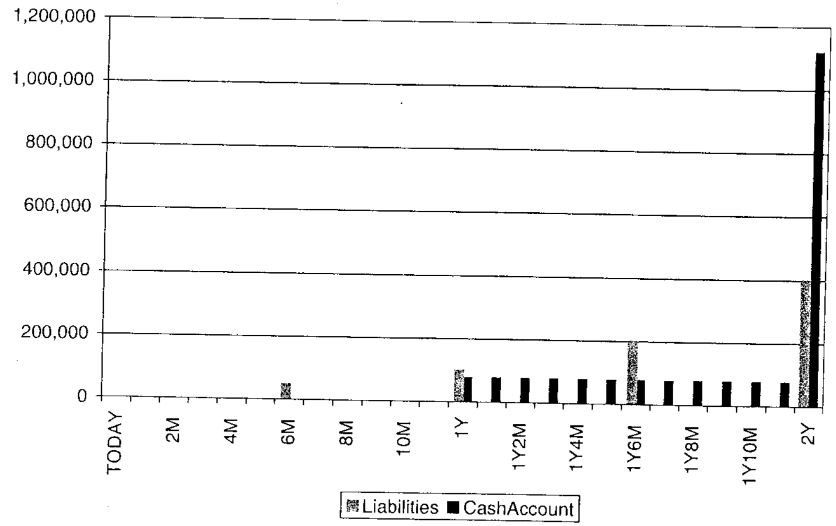

Figure 2: Value of the cash account in example 2 over time compared to the liability stream

This portfolio has an expected final value (including previous coupon payments) of 1,105,841 *Euro* corresponding to an expected rate of return of 5.03 %. The probabilities of falling below the benchmark C_0 at times $t \in \mathcal{T}_L$ are given by the vector (1%, 1%, 0%, 1%). To satisfy the shortfall constraints, the investment in the Greek bond is reduced and shifted to the less risky Italian and German bonds. Unfortunately, the liability stream is also not covered by this portfolio.

The optimal country allocation is GER: 44% , ITA: 25%, GRE: 31%, Cash: 0%. The cash account compared to the liability stream is shown in figure 2.

Example 3. In this example we add the liability constraints (but not the shortfall constraints of example 2) at times $t \in \mathcal{T}_L$ to the budget constraint of example 1 getting an optimal portfolio which consists of

7, 009 (725,703 *Euro*) of the 2 year 8% Greek bond
1, 852 (197,672 *Euro*) of the1.5 year 8% Greek bond
411 (41,440 *Euro*) of the 1 year 7% Italian bond
35,185 *Euro* on the cash account.

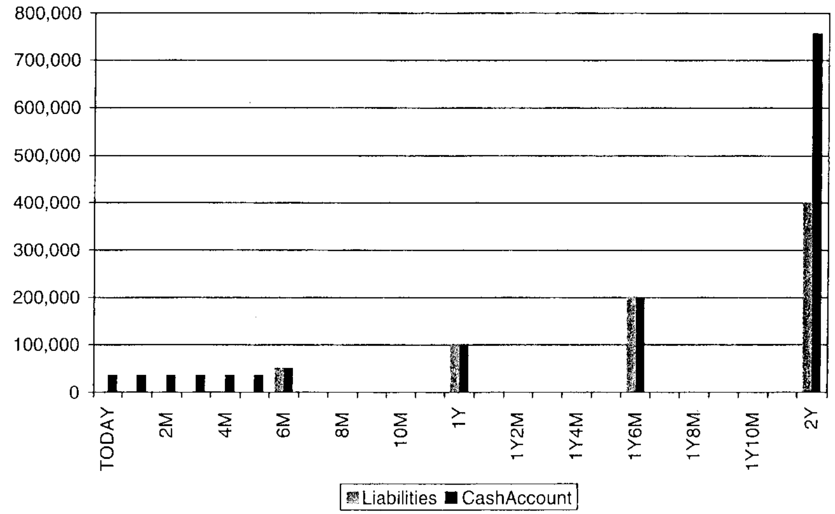

Figure 3: Value of the cash account in example 3 over time compared to the liability stream

This portfolio has an expected final value (including previous coupon payments) of 1,106,961 *Euro* corresponding to an expected rate of return of 5.08 %. It can be seen that the maturities of the bonds are chosen according to the liability payment dates $t \in \mathcal{T}_L$. The probabilities of falling below the benchmark C_0 at times $t \in \mathcal{T}_L$ are given by the vector (5%, 2%, 0%, 0%). Unfortunately, the shortfall constraints are not met by this portfolio. The optimal country allocation is GER: 0% , ITA: 4%, GRE: 92%, Cash: 4%. The optimal maturity allocation is 1Y: 4.14% , 1.5Y: 19.77%, 2Y: 72.57%, Cash: 3.52%.The value of the cash account compared to the liability stream is shown in figure 3.

Example 4. In this example we add the shortfall and liability constraints at times $t \in \mathcal{T}_L$ to the budget constraint of example 1 getting an optimal portfolio which consists of

3, 050 (315, 815 *Euro*)	of the 2 year 8%	Greek bond
2,133 (217,370 *Euro*)	of the 2 year 7%	Italian bond
1,869 (196,040 *Euro*)	of the1.5 year 7%	Italian bond
465 (46,900 *Euro*)	of the 1 year 7%	Italian bond
1,825 (186,959 *Euro*)	of the 2 year 6%	German bond
36,916 *Euro*	on the cash account.	

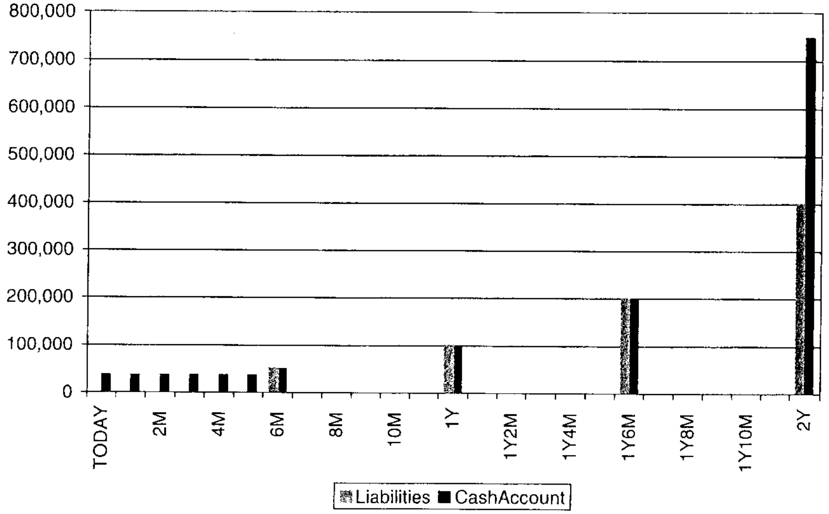

Figure 4: Value of the cash account in example 4 over time compared to the liability stream

This portfolio has an expected final value (including previous coupon payments) of 1,099,185 *Euro* corresponding to an expected rate of return of 4.73%. We now have a portfolio with maturity times according to the liability payment dates $t \in \mathcal{T}_L$ and a country diversification including Italian and German bonds to increase the safety of the portfolio, i.e. to reduce the probability of falling below the given benchmark. The probabilities of falling below the benchmark C_0 at times $t \in \mathcal{T}_L$ are given by the vector (1%, 1%, 0%, 0%). The optimal country allocation is GER: 19% , ITA: 45%, GRE: 32%, Cash: 4%. The optimal maturity allocation is 1Y: 4.69%, 1.5Y: 19.60%, 2Y: 72.01%, Cash: 3.59%. The value of the cash account compared to the liability stream is shown in figure 4.

6 Conclusion

We used the model of Schmid & Zagst (2000) for simultaneously pricing defaultable and non-defaultable sovereign bonds. The parameter estimations were done using Kalman filtering methods as described in (Schmid & Kalemanova 2002). Based on these estimates the prices of the bonds were simulated over time and the simulations were used to derive optimal portfolios consisting of both, defaultable and non-defaultable sovereign bonds. One set of restrictions was due to a minimum required liability stream, the second set of restrictions was to limit the potential of falling below some given benchmark. To visualize our methotology we presented a case study using German, Italian, and Greek sovereign bonds. Further research will consider correlations of the Wiener processes and the examination of emerging market and corporate bond portfolios.

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