The Normal Inverse Gaussian Distribution for Synthetic CDO Pricing

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Abstract

This paper presents an extension of the popular Large Homogeneous Portfolio (LHP) approach to the pricing of CDOs. LHP (which has already become a standard model in practice) assumes a flat default correlation structure over the reference credit portfolio and models defaults using a one factor Gaussian copula. However, this model fails to fit the prices of different CDO tranches simultaneously which leads to the well known implied correlation skew. Many researchers explain this phenomenon with the lack of tail dependence in the Gaussian copula and propose to use a Student t-distribution. Incorporating the effect of tail dependence into the one factor portfolio credit model yields significant pricing improvement. However, the computation time increases dramatically as the Student t-distribution is not stable under convolution. This makes it impossible to use the model for computationally intensive applications such as the determination of the optimal asset allocation in an investor's portfolio over different asset classes including CDOs. We present a modification of the LHP model replacing the Student t-distribution with the Normal inverse Gaussian (NIG) distribution. We compare the properties of our new model with those of the Gaussian and the double t-copulas. The employment of the NIG distribution not only speeds up

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the computation time significantly but also brings more flexibility into the dependence structure.

Introduction

The calculation of loss distributions of the portfolio of reference instruments over different time horizons is the central problem of pricing synthetic CDOs. The factor copula approach for modeling correlated defaults has become very popular. Unfortunately, computationally intensive Monte Carlo simulation techniques have to be used if the correlation structure is assumed to be completely general. However, the concept of conditional independence yields substantial simplification: If it is assumed that defaults of different names in the credit portfolio are independent conditional on a common market factor it is much simpler to compute the aggregate portfolio loss distributions for different time horizons. The factor (or conditional independence) approach allows use of semi-analytic computation techniques avoiding time consuming Monte Carlo simulations. Examples are the approaches described by Laurent and Gregory [2003] who use fast Fourier transform techniques, as well as Hull and White [2004] and Andersen, Sidenius and Basu [2003] who apply an iterative numerical procedure to build up the loss distribution for the pool of reference instruments.

Going one step further and making the additional simplifying assumption of a large homogeneous portfolio (LHP), i.e. assuming it is possible to approximate the real reference credit portfolio with a portfolio consisting of a large number of equally weighted identical instruments (having the same term structure of default probabilities, recovery rates, and correlations to the common factor), we get a closed-form analytic synthetic CDO pricing formula. This LHP limit approximation employing the law of large numbers was first proposed by Vasicek [1987; 1991; 2002].

In the case of the one factor Gaussian copula all integrals in the pricing formulas can be computed analytically (see e.g. O'Kane and Schloegl [2001]). Due to its simplicity, this model has become the market standard. However, there is a fundamental problem:

If we calculate the correlations that are implied by the market prices of tranches of the same CDO using the LHP approach, we do not get the same correlation over the whole structure but rather, we observe a correlation skew. The main explanation of this phenomenon is the lack of tail dependence of the Gaussian copula. Various authors have proposed different ways to bring more tail dependence into the model. One approach is the introduction of additional stochastic factors into the model. Andersen and Sidenius [2005] extended the Gaussian factor copula model to random recovery and random factor loadings. Trinh, Thompson and Devarajan [2005] allowed for idiosyncratic and systematic jumps to default. Many other authors proposed a different approach - to use a copula that exhibits more tail dependence: Examples are the Marshall-Olkin copula in Andersen and Sidenius [2005], the Student t-copula in O'Kane and Schloegl [2003], and the double t-distribution in Hull and White [2004].

Burtschell, Gregory and Laurent [2005] performed a comparative analysis of a Gaussian copula model, stochastic correlation extension to Gaussian copula, Student t-copula model, double t-factor model, Clayton copula and Marshall-Olkin copula. By pricing the tranches of DJ iTraxx they showed that Student t and Clayton copula models provided results very similar to the Gaussian copula model. The Marshall-Olkin copula led to a dramatic fattening of the tail. The results of the double t-factor model and stochastic correlation model were closer to the market quotes, and the factor loading model of Andersen and Sidenius [2005] performed similarly to the latter ones.

Unfortunately, the integrals in the synthetic CDO pricing formula in the LHP model that is based on the double t-copula cannot be computed analytically. The major problem is the instability of the Student t-distribution under convolution. The calculation of the default thresholds (that are quantiles of the distribution of asset returns) requires a numerical root search procedure involving numerical integration that increases the computation time dramatically (see section 3). Thus, finding a different heavy tailed distribution that is similar to the Student t but stable under convolution could decrease the computation time tremendously. As computation time is an important issue for a large range of applications such as the determination of an optimal portfolio asset allocation (including CDO tranches) where CDO tranches have to be repriced in each scenario path at each time step in the future, the use of such a distribution is crucial.

In our opinion, the Normal Inverse Gaussian (NIG) distribution is an appropriate distribution to solve the problem. The family of NIG distributions is a special case of the generalized hyperbolic distributions (Barndorff-Nielsen [1978]). Due to their specific characteristics, NIG distributions are very interesting for applications in finance - they are a generally flexible four parameter distribution family that can produce fat tails and skewness, the class is convolution stable under certain conditions and the cumulative distribution function, density and inverse distribution functions can still be computed sufficiently fast (Kalemanova and Werner [2006]). The distribution has been employed for stochastic volatility modeling, e.g., Barndorff-Nielsen [1997].

This paper is organized as follows: In the first section we recall the general pricing approach for synthetic CDOs. The one factor Gaussian copula model and the LHP approach developed by Vasicek [1987; 1991; 2002] are presented in the second section. In the third section we describe the extension of LHP to a double t-factor model proposed by Hull and White [2004]. The fourth section contains a brief review of the properties of the NIG distribution. In the fifth section we present the one factor NIG copula model and the CDO pricing formulas for our NIG LHP model. Finally, the last two sections provide an empirical investigation and comparative analysis of the pricing abilities of the three models and their tail dependence.

1 General semi-analytic approach for pricing synthetic CDOs

We consider a synthetic CDO with a reference portfolio consisting of credit default swaps only. A protection seller of a synthetic CDO tranche receives from the protection buyer spread payments on the outstanding notional principal at regular payment dates (usually quarterly). If the total loss of the reference credit portfolio exceeds the notionals of the subordinated tranches, the protection seller has to make compensation payments for these losses to the protection buyer.

Basically, the pricing of a synthetic CDO tranche that takes losses from K_1 to K_2 (with $0 \leq K_1 < K_2 \leq 1$) of the reference portfolio works in the same way as the pricing of a credit default swap. Let's assume that $t_1 < ... < t_n = T$ denote the spread payment dates with T the maturity of the synthetic CDO. Further, t_0 such that $t_0 < t_1$ is the valuation date. More precisely, the premium payments are made in arrears - at time t_k for the payment period from t_{k-1} to t_k. For simplicity we assume that the premium at time t_k is paid on the notional outstanding at this point of time.

Now we introduce some further notation. We denote the annual spread that is the base for the calculation of the premium payments as s. $L_{(K_1,K_2)}(t)$ denotes the pro-rata loss of the tranche (K_1, K_2) up to time t. $r(t)$ is the short rate that is assumed to be independent from the tranche loss. We consider the risk-neutral measure Q and denote the expected tranche loss $E_Q\left[L_{(K_1,K_2)}(t)\right]$ as $EL_{(K_1,K_2)}(t)$ and the discount factor $E_Q\left[e^{-\int_{t_0}^{t_i} r(u)du}\right]$ as $B(t_0, t_i)$.

The value of the premium leg of the tranche is computed as the present value of all expected spread payments:

$$\begin{aligned} \text{Premium Leg} &= \sum_{i=1}^{n} \Delta t_i \cdot s \cdot E_Q\left[\left(1 - L_{(K_1,K_2)}(t_i)\right) e^{-\int_{t_0}^{t_i} r(u)du}\right] \\ &= \sum_{i=1}^{n} \Delta t_i \cdot s \cdot \left(1 - EL_{(K_1,K_2)}(t_i)\right) \cdot B(t_0, t_i), \end{aligned} \tag{1}$$

where $\Delta t_i = t_i - t_{i-1}$.

Protection payments are made immediately at default. However, for simplicity we assume that the protection is paid only at times t_k as well. The protection payment at time t_k equals the notional amount of the tranche that defaulted during the previous payment period. Then the value of the protection leg can be calculated according to:

$$
\begin{aligned}
\text{Protection Leg} &= E_Q\left[\int_{t_0}^{t_n} e^{-\int_{t_0}^{s} r(u)du} dL_{(K_1,K_2)}(s)\right] \\
&\approx \sum_{i=1}^{n} E_Q\left[e^{-\int_{t_0}^{t_i} r(u)du}\left(L_{(K_1,K_2)}(t_i) - L_{(K_1,K_2)}(t_{i-1})\right)\right] \\
&= \sum_{i=1}^{n}\left(EL_{(K_1,K_2)}(t_i) - EL_{(K_1,K_2)}(t_{i-1})\right)\cdot B(t_0,t_i). \qquad (2)
\end{aligned}
$$

At issuance of the CDO tranche the tranche spread is determined so that the values of premium leg and protection leg are equal:

$$
s = \frac{\sum_{i=1}^{n}\left(EL_{(K_1,K_2)}(t_i) - EL_{(K_1,K_2)}(t_{i-1})\right)\cdot B(t_0,t_i)}{\sum_{i=1}^{n}\Delta t_i\cdot\left(1 - EL_{(K_1,K_2)}(t_i)\right)\cdot B(t_0,t_i)}. \qquad (3)
$$

Given the portfolio loss $L_{portfolio}(t)$, the corresponding percentage tranche loss is calculated as

$$
L_{(K_1,K_2)}(t) = \frac{(\min(L_{portfolio}(t),K_2) - K_1)^+}{K_2 - K_1}. \qquad (4)
$$

Assume, a continuous portfolio loss distribution function $F(t,x)$ is known. Then the percentage expected loss of the CDO tranche (K_1,K_2) can be computed as:

$$
EL_{(K_1,K_2)}(t) = \frac{1}{K_2 - K_1}\int_{K_1}^{1}(\min(x,K_2) - K_1)dF(t,x). \qquad (5)
$$

The expected tranche loss in Equation (5) can also be written as

$$
EL_{(K_1,K_2)}(t) = \frac{1}{K_2 - K_1}\left(\int_{K_1}^{1}(x - K_1)dF(t,x) - \int_{K_2}^{1}(x - K_2)dF(t,x)\right). \qquad (6)
$$

Thus, the central problem in the pricing of a CDO tranche is to derive the loss distribution of the reference portfolio.

In the next sections we present the factor copula model of correlated defaults as well as an analytical approximation method to compute the portfolio loss distribution and the expected loss of a tranche.

2 One factor Gaussian model and LHP approximation

The LHP approach is based on a one factor Gaussian copula model of correlated defaults. Assume that the portfolio of reference assets consists of m financial instruments and the asset return until time t of the i-th issuer in the portfolio, $A_i(t)$, is assumed to be of the form:

$$A_i(t) = a_i M(t) + \sqrt{1 - a_i^2} X_i(t), \tag{7}$$

where $M(t), X_i(t), i = 1, ..., m$, are independent standard normally distributed random variables. Then, conditionally on the common market factor $M(t)$, the asset returns of the different issuers are independent. Note, that due to the stability of normal distributions under convolution the asset return $A_i(t)$ follows a standard normal distribution as well.

Under this copula model the variable $A_i(t)$ is mapped to default time t_i of the i-th issuer using a percentile-to-percentile transformation, i.e.

$$P[t_i \leq t] = P[A_i(t) \leq C_i(t)]. \tag{8}$$

We denote the probability of the issuer i to default before time t with

$$q_i(t) = P[t_i \leq t]. \tag{9}$$

The risk-neutral probabilities are implied from the observable market prices of credit default instruments (e.g. bonds or CDS).

Then the thresholds $C_i(t)$ can be computed as

$$C_i(t) = \Phi^{-1}(q_i(t)), \tag{10}$$

where Φ is the standard normal distribution function.

According to Equation (7), the i-th issuer defaults by time t if

$$X_i(t) \leq \frac{C_i(t) - a_i M(t)}{\sqrt{1 - a_i^2}}. \tag{11}$$

Then the probability that the i-th issuer defaults by time t conditional on the factor $M(t)$ is

$$p_i(t|M) = \Phi\left(\frac{C_i(t) - a_i M(t)}{\sqrt{1 - a_i^2}}\right). \tag{12}$$

If we assume that the portfolio is homogeneous, i.e. $a_i = a$ and $C_i(t) = C(t)$ for all i and the notional amounts and recovery R are the same for all issuers, then the default probability of all issuers in the portfolio conditional on M is given by

$$p(t|M) = \Phi\left(\frac{C(t) - aM(t)}{\sqrt{1 - a^2}}\right). \tag{13}$$

For simplicity let us first assume a zero recovery rate. Then the percentage portfolio loss $L(t)$ takes values $\left(\frac{k}{m}\right)_{k=0,\ldots,m}$ with probability

$$P\left[L(t) = \frac{k}{m}|M(t)\right] = \begin{pmatrix} m \\ k \end{pmatrix} \Phi\left(\frac{C(t) - aM(t)}{\sqrt{1 - a^2}}\right)^k \left(1 - \Phi\left(\frac{C(t) - aM(t)}{\sqrt{1 - a^2}}\right)\right)^{m-k}. \tag{14}$$

Due to conditional independence and only two possible states the conditional loss distribution is binomial. The unconditional loss distribution $P\left[L(t) = \frac{k}{m}\right]$ can be obtained by integrating Equation (14) with the distribution of the factor $M(t)$:

$$P\left[L(t) = \frac{k}{m}\right] = \int_{-\infty}^{\infty} \begin{pmatrix} m \\ k \end{pmatrix} \Phi\left(\frac{C(t) - au}{\sqrt{1 - a^2}}\right)^k \left(1 - \Phi\left(\frac{C(t) - au}{\sqrt{1 - a^2}}\right)\right)^{m-k} d\Phi\left(u\right). \tag{15}$$

Since the calculation of the loss distribution in Equation (14) is quite computationally intensive for large m, it is desirable to use some approximation. The large portfolio limit approximation proposed by Vasicek [2002] is a very simple but powerful method.

We consider the cumulative probability of the percentage portfolio loss not exceeding x for x in $[0,1]$:

$$F_m(t,x) = \sum_{k=0}^{[mx]} P\left[L(t) = \frac{k}{m}\right]. \tag{16}$$

Substituting $s = \Phi\left(\frac{C(t)-au}{\sqrt{1-a^2}}\right)$ and plugging in Equation (15) we get the following expression for $F_m(t,x)$:

$$F_m(t,x) = -\int_0^1 \sum_{k=0}^{[mx]} \begin{pmatrix} m \\ k \end{pmatrix} s^k (1-s)^{m-k} d\Phi\left(\frac{C(t)-\sqrt{1-a^2}\Phi^{-1}(s)}{a}\right). \tag{17}$$

Now we take it to the limit and consider the portfolio loss distribution function of an infinitely large portfolio:

$$F_\infty(t,x) = \lim_{m\to\infty}\left[-\int_0^1 \sum_{k=0}^{[mx]} \begin{pmatrix} m \\ k \end{pmatrix} s^k (1-s)^{m-k} d\Phi\left(\frac{C(t)-\sqrt{1-a^2}\Phi^{-1}(s)}{a}\right)\right]. \tag{18}$$

Since

$$\lim_{m\to\infty}\sum_{k=0}^{[mx]} \begin{pmatrix} m \\ k \end{pmatrix} s^k (1-s)^{m-k} = \begin{cases} 0, & \text{if } x < s \\ 1, & \text{if } x > s \end{cases} \tag{19}$$

the cumulative distribution function of losses of a large portfolio equals

$$
\begin{aligned}
F_\infty(t,x) &= -\int_0^x d\Phi\left(\frac{C(t)-\sqrt{1-a^2}\Phi^{-1}(s)}{a}\right) \\
&= -\Phi\left(\frac{C(t)-\sqrt{1-a^2}\Phi^{-1}(x)}{a}\right)+1. \qquad (20)
\end{aligned}
$$

Due to symmetry of the Gaussian distribution we can rewrite Equation (20) as:

$$
F_\infty(t,x) = \Phi\left(\frac{\sqrt{1-a^2}\Phi^{-1}(x)-C(t)}{a}\right). \qquad (21)
$$

Therefore, in case of the large homogeneous portfolio assumption it is possible to compute the integrals in Equation (5) analytically and the expected loss of the tranche is given by:

$$
EL_{(K_1,K_2)}(t) = \frac{\Phi_2\left(-\Phi^{-1}\left(K_1\right), C(t), \rho\right) - \Phi_2\left(-\Phi^{-1}\left(K_2\right), C(t), \rho\right)}{K_2-K_1},
$$

where Φ_2 is the bivariate normal distribution function and the covariance matrix

$$
\rho = \begin{pmatrix} 1 & -\sqrt{1-a^2} \\ -\sqrt{1-a^2} & 1 \end{pmatrix}. \qquad (22)
$$

Now, let us assume that assets have the same (maybe non-zero) recovery rate R. Then, the total loss of the equity tranche of K will occur only when assets of the total amount of $\frac{K}{1-R}$ have defaulted. Thus, the expected loss of the tranche between K and 1 is given by

$$
\begin{aligned}
&\int_{\frac{K}{1-R}}^{1} (1-R)\left(x-\frac{K}{1-R}\right) dF_\infty(t,x) \\
&= (1-R)\,\Phi_2\left(-\Phi^{-1}\left(\frac{K}{1-R}\right), C(t), \rho\right). \qquad (23)
\end{aligned}
$$

Finally, it is easy to see that the expected percentage loss of the mezzanine tranche taking losses from K_1 to K_2 under the assumption of constant recovery R is

$$
EL^R_{(K_1,K_2)}(t) = EL_{\left(\frac{K_1}{1-R},\frac{K_2}{1-R}\right)}(t). \qquad (24)
$$

3 LHP with double t-copula

One natural extension of the LHP approach is to use a distributional assumption that produces heavy tail. The double t one factor model proposed by Hull and White [2004] assumes Student t-distributions for the common market factor $M(t)$ as well as for the individual factors $X_i(t)$. Then the loss distribution $F_\infty(t,x)$ in Equation (21) becomes:

$$F_\infty(t,x) = T\left(\frac{\sqrt{1-a^2}T^{-1}(x) - C(t)}{a}\right), \tag{25}$$

where T denotes the Student t-distribution function. Unfortunately, it is not possible to solve the integral in Equation (5) analytically using this loss distribution and one has to use some numerical integration method.

The asset returns $A_i(t)$ do not follow necessarily Student t-distributions since the Student t-distribution is not stable under convolution. The distribution function H_i of $A_i(t)$ must be computed numerically. Afterwards, it is possible to calculate the default thresholds $C_i(t)$ by $H_i^{-1}(q_i(t))$. This procedure is quite time consuming and it makes the double t-model too slow for Monte Carlo based risk management applications.

4 NIG distribution and its main properties

The normal inverse Gaussian distribution is a mixture of normal and inverse Gaussian distributions.

A non-negative random variable Y has an Inverse Gaussian (IG) distribution with parameters $\alpha > 0$ and $\beta > 0$ if its density function is of the form:

$$f_{\mathcal{IG}}(y;\,\alpha,\,\beta) = \begin{cases} \dfrac{\alpha}{\sqrt{2\pi\beta}}\, y^{-3/2}\, \exp\left(-\dfrac{(\alpha-\beta y)^2}{2\beta y}\right) & ,\ \text{if } y > 0 \\ 0 & ,\ \text{if } y \le 0. \end{cases} \tag{26}$$

We write then $Y \sim \mathcal{IG}(\alpha,\,\beta)$.

A random variable X follows a Normal Inverse Gaussian (NIG) distribution with parameters α, β, μ and δ if:

$$\begin{aligned} X \mid Y = y &\sim \mathcal{N}(\mu + \beta y,\, y) \\ Y &\sim \mathcal{IG}(\delta\gamma,\, \gamma^2) \text{ with } \gamma := \sqrt{\alpha^2 - \beta^2}, \end{aligned} \tag{27}$$

with parameters satisfying the following conditions: $0 \leq |\beta| < \alpha$ and $\delta > 0$. We then write $X \sim \mathcal{NIG}(\alpha,\, \beta,\, \mu,\, \delta)$ and denote the density and probability functions by $f_{\mathcal{NIG}}(x;\, \alpha,\, \beta,\, \mu,\, \delta)$ and $F_{\mathcal{NIG}}(x;\, \alpha,\, \beta,\, \mu,\, \delta)$ correspondingly.

The density of a random variable $X \sim \mathcal{NIG}(\alpha,\, \beta,\, \mu,\, \delta)$ is given by:

$$f_{\mathcal{NIG}}(x;\, \alpha,\, \beta,\, \mu,\, \delta) = \frac{\delta\alpha \cdot \exp\left(\delta\gamma + \beta(x-\mu)\right)}{\pi \cdot \sqrt{\delta^2 + (x-\mu)^2}} K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right), \tag{28}$$

where $K_1(w) := \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}w(t + t^{-1})\right) dt$ is the modified Bessel function of the third kind.

While the density function of NIG distribution is quite complicated, its moment generating function has a simple form. The NIG moment generating function $M(t) = E[\exp(tx)]$ is given by:

$$M_{\mathcal{NIG}}(x;\, \alpha,\, \beta,\, \mu,\, \delta) = \exp(\mu t)\, \frac{\exp\left(\delta\sqrt{\alpha^2 - \beta^2}\right)}{\exp\left(\delta\sqrt{\alpha^2 - (\beta + t)^2}\right)}. \tag{29}$$

The main properties of the NIG distribution class are the scaling property

$$X \sim \mathcal{NIG}(\alpha,\, \beta,\, \mu,\, \delta) \Rightarrow cX \sim \mathcal{NIG}\left(\frac{\alpha}{c},\, \frac{\beta}{c},\, c\mu,\, c\delta\right), \tag{30}$$

and the closure under convolution for independent random variables X and Y

$$X \sim \mathcal{NIG}(\alpha,\, \beta,\, \mu_1,\, \delta_1), Y \sim \mathcal{NIG}(\alpha,\, \beta,\, \mu_2,\, \delta_2)$$
$$\Rightarrow X + Y \sim \mathcal{NIG}\left(\alpha,\, \beta,\, \mu_1 + \mu_2,\, \delta_1 + \delta_2\right). \tag{31}$$

The central moments (mean, variance, skewness and kurtosis) of a random variable $X \sim \mathcal{NIG}(\alpha,\, \beta,\, \mu,\, \delta)$ are:

$$\mathbb{E}(X) = \mu + \delta\frac{\beta}{\gamma} \qquad \mathbb{V}(X) = \delta\frac{\alpha^2}{\gamma^3}$$
$$\mathbb{S}(X) = 3\frac{\beta}{\alpha \cdot \sqrt{\delta\gamma}} \qquad \mathbb{K}(X) = 3 + 3\left(1 + 4\left(\frac{\beta}{\alpha}\right)^2\right)\frac{1}{\delta\gamma}.$$

Normal Inverse Gaussian distribution usually does not belong to the package of standard distributions that are already implemented in programs like Matlab, S-Plus, R and Mathematica. Since the NIG distribution functions are quite complicated we would expect them to be computationally intensive if using the straight forward implementation. The work in Kalemanova and Werner [2006] showed that this is indeed the case. We have developed and compared some alternative implementations of NIG distribution in Matlab. Since the computational speed is especially important to us, we spent some additional efforts to make the implementation very efficient. Our NIG toolbox can be downloaded from Matlab central file exchange.

5 One factor NIG copula model and LHP approximation

Now we want to apply NIG distribution to the one factor copula model of correlated defaults and to derive the semi-analytic pricing formulas for the large homogeneous portfolio under the NIG copula model. Since the convolution property in Equation (31) of the NIG distribution does not hold for two arbitrary NIG random variables we need to find the right parametrization of the factors M and X_i in the copula model so that A_i follows NIG distribution as well.

We start with $M \sim \mathcal{NIG}(\alpha_1,\, \beta_1,\, \mu_1,\, \delta_1)$ and $X_i \sim \mathcal{NIG}(\alpha_2,\, \beta_2,\, \mu_2,\, \delta_2)$. Then applying the scaling property (30) we get:

$$aM \sim \mathcal{NIG}\left(\frac{\alpha_1}{a}, \frac{\beta_1}{a}, a\mu_1, a\delta_1\right) \tag{32}$$

$$\sqrt{1-a^2}X_i \sim \mathcal{NIG}\left(\frac{\alpha_2}{a}, \frac{\beta_2}{a}, a\mu_2, a\delta_2\right). \tag{33}$$

Further, to be able to apply the convolution property to the expression $aM + \sqrt{1-a^2}X_i$, the two first parameters in (32) and (33) must be equal, i.e.

$$\frac{\alpha_1}{a} = \frac{\alpha_2}{a} \tag{34}$$

$$\frac{\beta_1}{a} = \frac{\beta_2}{a}. \tag{35}$$

Since M is the common market factor is should not depend on the portfolio correlation parameter a. So we set

$$\alpha_1 = \alpha \tag{36}$$

$$\beta_1 = \beta \tag{37}$$

$$\alpha_2 = \frac{\sqrt{1-a^2}}{a}\alpha \tag{38}$$

$$\beta_2 = \frac{\sqrt{1-a^2}}{a}\beta. \tag{39}$$

Now the random variable $A_i = aM + \sqrt{1-a^2}X_i$ is NIG distributed for any μ_1, μ_2, δ_1 and δ_2. Its parameters are:

$$A_i \sim \mathcal{NIG}\left(\frac{\alpha}{a}, \frac{\beta}{a}, a\mu_1 + \sqrt{1-a^2}\mu_2, a\delta_1 + \sqrt{1-a^2}\delta_2\right). \tag{40}$$

Next, we restrict the parameters further in order to standardize the distributions of the both factors. The third and the fourth parameters are chosen so that the distributions have zero mean and unit variance:

$$\mu_1 = -\frac{\beta\gamma^2}{\alpha^2} \tag{41}$$

$$\delta_1 = \frac{\gamma^3}{\alpha^2} \tag{42}$$

$$\mu_2 = -\frac{\sqrt{1-a^2}}{a}\frac{\beta\gamma^2}{\alpha^2} \tag{43}$$

$$\delta_2 = \frac{\sqrt{1-a^2}}{a}\frac{\gamma^3}{\alpha^2}, \tag{44}$$

with $\gamma = \sqrt{\alpha^2 - \beta^2}$.

Then, the distribution of A_i is:

$$A_i \sim \mathcal{NIG}\left(\frac{\alpha}{a}, \frac{\beta}{a}, -\frac{1}{a}\frac{\beta\gamma^2}{\alpha^2}, \frac{1}{a}\frac{\gamma^3}{\alpha^2}\right). \tag{45}$$

Note that it has zero mean and unit variance as well.

Now we can summarize the obtained results and define the one factor NIG copula model.

Consider a homogeneous portfolio of m credit instruments. The standardized asset return up to time t of the i-th issuer in the portfolio, $A_i(t)$, is assumed to be of the form:

$$A_i(t) = aM(t) + \sqrt{1-a^2}X_i(t), \tag{46}$$

with independent random variables

$$M(t) \sim \mathcal{NIG}\left(\alpha, \beta, -\frac{\beta\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}\right) \tag{47}$$

$$X_i(t) \sim \mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\alpha, \frac{\sqrt{1-a^2}}{a}\beta, -\frac{\sqrt{1-a^2}}{a}\frac{\beta\gamma^2}{\alpha^2}, \frac{\sqrt{1-a^2}}{a}\frac{\gamma^3}{\alpha^2}\right), \tag{48}$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$.

Under this copula model the variable $A_i(t)$ is mapped to default time t_i of the i-th issuer using a percentile-to-percentile transformation, i.e.

$$P[t_i \leq t] = P[A_i(t) \leq C(t)]. \tag{49}$$

To simplify notation we denote the distribution function $F_{\mathcal{NIG}}\left(x;\, s\alpha,\, s\beta,\, -s\frac{\beta\gamma^2}{\alpha^2},\, s\frac{\gamma^3}{\alpha^2}\right)$ with $F_{\mathcal{NIG}(s)}(x)$. So, for example, the distribution function of the factor M is $F_{\mathcal{NIG}(1)}(x)$ and of the factor X_i $F_{\mathcal{NIG}(\frac{\sqrt{1-a^2}}{a})}(x)$.

Now consider an infinitely large homogeneous portfolio with the asset returns following a one factor NIG copula model. Then the distribution of the portfolio loss is given by

$$F_\infty(t,x) = 1 - F_{\mathcal{NIG}(1)}\left(\frac{C(t) - \sqrt{1-a^2}F^{-1}_{\mathcal{NIG}(\frac{\sqrt{1-a^2}}{a})}(x)}{a}\right), \tag{50}$$

with $x \in [0,1]$ the percentage portfolio loss and $C(t) = F^{-1}_{\mathcal{NIG}(\frac{1}{a})}(q(t))$, where $q(t)$ is the risk-neutral default probability of each issuer in the portfolio.

To compute the tranche expected loss in the one factor NIG model we use expression (6) and rewrite it as

$$EL_{(K_1,K_2)}(t) = \frac{1}{K_2 - K_1}\int_{K_1}^{K_2} (x - K_1)dF_\infty(t,x) + (1 - F_\infty(t,K_2))\,. \tag{51}$$

To compute the integral we need the density function of the portfolio loss:

$$f_\infty(t,x) = \frac{\sqrt{1-a^2}}{a}\frac{f_{\mathcal{NIG}(1)}\left(\frac{C(t)-\sqrt{1-a^2}F^{-1}_{\mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\right)}(x)}{a}\right)}{f_{\mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\right)}\left(F^{-1}_{\mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\right)}(x)\right)}. \tag{52}$$

The integral

$$\int_{K_1}^{K_2} (x - K_1) f_\infty(t,x) dx \tag{53}$$

has no analytical solution and has to be computed numerically. The inverse distribution function of the NIG distribution is quite computationally intensive. Computing this integral numerically involves the evaluation of the inverse distribution function numerous times. However, it is very easy to avoid this by eliminating it by means of the variable change:

$$y = F^{-1}_{\mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\right)}(x). \tag{54}$$

Then

$$dy = \frac{dx}{f_{\mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\right)}\left(F^{-1}_{\mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\right)}(x)\right)}. \tag{55}$$

So we get

$$\int_{K_1}^{K_2} (x - K_1) f_\infty(t,x) dx \tag{56}$$

$$= \int_{F^{-1}_{\mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\right)}(K_1)}^{F^{-1}_{\mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\right)}(K_2)} \left(F_{\mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\right)}(y) - K_1\right) f_{\mathcal{NIG}(1)}\left(\frac{C(t) - \sqrt{1-a^2}y}{a}\right) \frac{\sqrt{1-a^2}}{a} dx.$$

This expression contains the inverse NIG distribution function only in the integration limits. The function under the integral contains only NIG distribution function and density that are much faster to compute than the inverse distribution function.

The important advantage of the NIG copula model is that the default thresholds $C(t)$ are easy and fast to compute due to the convolution property of the NIG distribution.

6 Comparison of the models: pricing DJ iTraxx

In order to compare the properties of the LHP model with Gaussian, double t and NIG copulas, we consider the example of the tranched Dow Jones iTraxx Europe with 5 years maturity. The reference portfolio consists of 125 credit default swap names. The standard tranches have attachment/detachment points at 3%, 6%, 9%, 12% and 22%. The investors in the tranches receive quarterly spread payments on the outstanding notional and compensate for losses when these hit the tranche they are invested in. The investor in the equity tranche receives an up-front fee that is quoted in the market and an annual spread of 500 bp paid quarterly. The settlement date of this series of this index is 20-March-2006 and maturity 20-June-2011. We consider the market quotes of iTraxx tranches at 12-April-2006. The average CDS spread of the corresponding CDS portfolio is 32 bp on this day. We use the constant default intensity model to derive the marginal default distributions, and estimate the default intensity of the large homogeneous portfolio from the average portfolio CDS spread. The constant recovery rate is assumed to be 40%.

Gaussian and double t-factor copulas have only one parameter, the correlation. We estimate this parameter so that the price of the equity tranche fits the market quote, i.e. we calculate the implied equity correlation. The same correlation is used to price the other tranches. The versions of the NIG factor copula we consider have one parameter α ($\beta = 0$) or two parameters α and β besides the correlation. We minimize the sum of the absolute errors over all tranches to estimate these parameters.

Exhibit 1 presents the market quotes of the iTraxx tranches as well as the prices of the LHP model with Gaussian one factor copula, double t-distribution with 3 and 4 degrees of freedom and NIG factor copula with one and two free parameters. In the one parameter NIG copula, the parameter β is set to zero which makes the distribution symmetric. The double t-factor copula model fits only the equity tranche exactly since it has only one continuous valued parameter (correlation). The double t-model with 3 degrees of freedom underprices the second tranche while the double t model with 4 degrees of freedom overprices it. Since the second model parameter (degrees of freedom) is only integer valued it is in general impossible to fit the second tranche exactly. The results of the NIG copulas are similar to the results of double t-copulas. The additional free parameter in the NIG copula makes it more flexible: the second tranche can be fitted exactly as well. Surprisingly, one more free parameter β doesn't bring much improvement to the fitting results in this example. The NIG models overprice the three most senior tranches similarly to the double t-model. The overall results of the NIG models are slightly better then those of the double t-models. However, the important advantage of the NIG model is the much faster computation time.

We plot density and cumulative distribution functions of portfolio losses from the LHP model with Gaussian, double t(3) and NIG factor copulas in Exhibit 2. Exhibit 3 shows the differences between modified (t or NIG) and Gaussian densities. It is especially easy to see that all modified models redistribute risk out of the lower end of the equity tranche to its higher end. Note that the total risk difference within the equity tranche is zero since we have calibrated all models to fit the equity tranche. The Gaussian model allocates more risk to mezzanine tranches than the modified models do. Exhibit 4 shows that the NIG(2) copula allocates slightly more risk at the 3-6% tranche than the double t(3)-copula does. Exhibit 5 presents the density function of the asset returns. The density of the NIG(2) model is slightly skewed to the left.

We have reformulated the prices produced by all models into base correlations (Exhibit 6). Base correlations introduced by McGinty, Ahluwalia, Watts and Beinstein [2004] of JP Morgan are the implied Gaussian model correlations of the corresponding equity tranches, i.e. of 0-3%, 0-6%, 0-9%, 0-12% and 0-22% tranches. The NIG model with two parameters fits the first two market base correlations exactly.

Base correlation is very popular among market participants since due to its monotonicity it can be easily interpolated and used to price the off-market tranches with the Gaussian model. However, the big problem with this method is that it does not guarantee an arbitrage free pricing of different tranches. Willemann [2004] analyzes the properties of base correlations and shows some further drawbacks of the model. Also O'Kane and Livesey [2004] discuss the problems of the base correlations and explain that this is not a proper model but just a fix of the Gaussian copula model. The big advantage of our NIG model is that due to its better fit of the market quotes we can compute the price of any off-market tranche with the same correlation value that guarantees arbitrage-free pricing.

7 Comparison of the tail dependence

In this section we investigate the tail dependence of the one factor copulas under study. In particular, the amount of dependence in the lower-left-quadrant is relevant for modeling credit portfolios.

Let X_1 and X_2 be continuous random variables. We consider the coefficient of lower tail dependence:

$$\lambda_L(x) = P\{X_2 \leq x | X_1 \leq x\}. \tag{57}$$

Random variables X_1 and X_2 are said to be asymptotically dependent in the lower tail if $\lambda_L > 0$ and asymptotically independent in the lower tail if $\lambda_L = 0$, where

$$\lambda_L = \lim_{x \to -\infty} \lambda_L(x). \tag{58}$$

The coefficient of upper tail dependence is defined as

$$\lambda_U(x) = P\{X_2 > x | X_1 > x\}, \tag{59}$$

and X_1 and X_2 are asymptotically dependent in the upper tail if $\lambda_U > 0$, where

$$\lambda_U = \lim_{x \to \infty} \lambda_U(x). \tag{60}$$

We plot the tail dependence coefficients of Gaussian, double t(3) and NIG factor copulas in Exhibit 7. The Gaussian copula shows no tail dependence. The tail dependence coefficient of double t(3)-copula is significantly larger than that of the Gaussian copula.

The tail dependence of the NIG copula with one parameter is slightly lower than the tail dependence of the double t(3)-copula. Note, that the upper and the lower tail dependence structures of Gaussian, double t(3) and NIG(1) copulas are symmetric. This is not the case for the two parameter NIG copula. This copula has a higher lower tail dependence coefficient similar to that of the NIG(1) copula, and a very small upper tail dependence coefficient.

These properties of the copulas under study can be observed in Exhibit 8 as well. We have simulated pairs of correlated asset returns with Gaussian, double t with 3 degrees of freedom, NIG with one and two parameters factor copulas and plotted the contours of their joint densities. The double t-factor copula produces more extreme values than the other copulas. The NIG copula with two parameters has more extreme values in the lower left tail than in the upper right tail.

Conclusion

This paper has introduced the NIG factor copula as an extension of the LHP model for pricing synthetic CDOs. We have presented an analytical formula for the distribution function of the portfolio loss. In general, the NIG distribution has four free parameters. The standardized symmetric NIG distribution with zero mean and unit standard deviation still has one free real parameter. The ability of this structure to fit CDO tranches is similar to that of the double t-copula that has one free integer parameter. However, NIG can fit the second tranche exactly which double t in general cannot do. Furthermore, we have studied the properties of the one factor copula model with a skewed NIG distribution having two free parameters. This copula has non symmetric upper and lower tail dependence. The two parameter NIG factor copula model could bring only a very slight improvement in our example.

The main purpose of developing the NIG factor copula was simplification and speeding up computation of the default thresholds. Incorporating the effect of tail dependence into this simple one factor credit portfolio model by means of replacing the Gaussian distribution with a Student t-distribution for the factors has the following disadvantage: it is not possible to compute the distribution function of asset returns analytically because of the lack of stability under convolution. This leads to a dramatic increase in computation time and makes it impossible to use this model for some important applications in practice. The NIG distribution still has a higher tail dependence than the Gaussian distribution. In addition, it is stable under convolution under certain conditions. The NIG distribution not only brings significant improvement with respect to computation times but also gives more flexibility in the modeling of the dependence structure of a credit portfolio.

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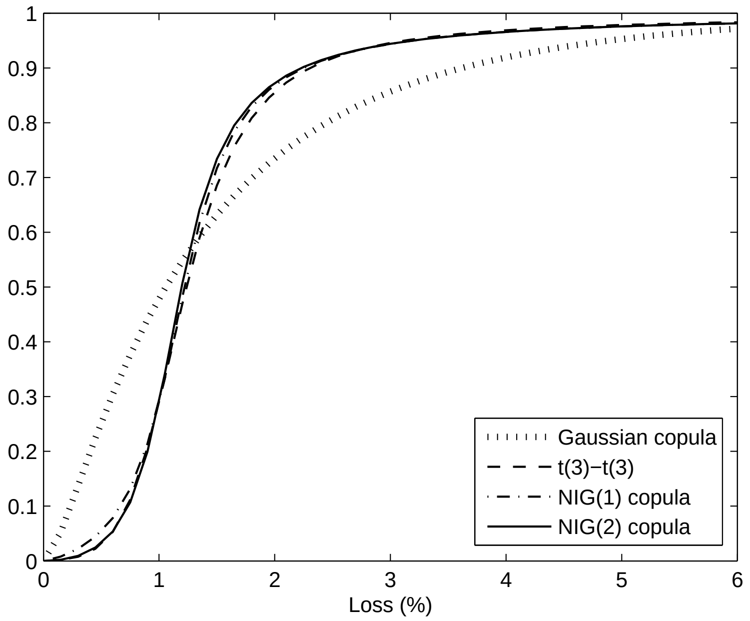
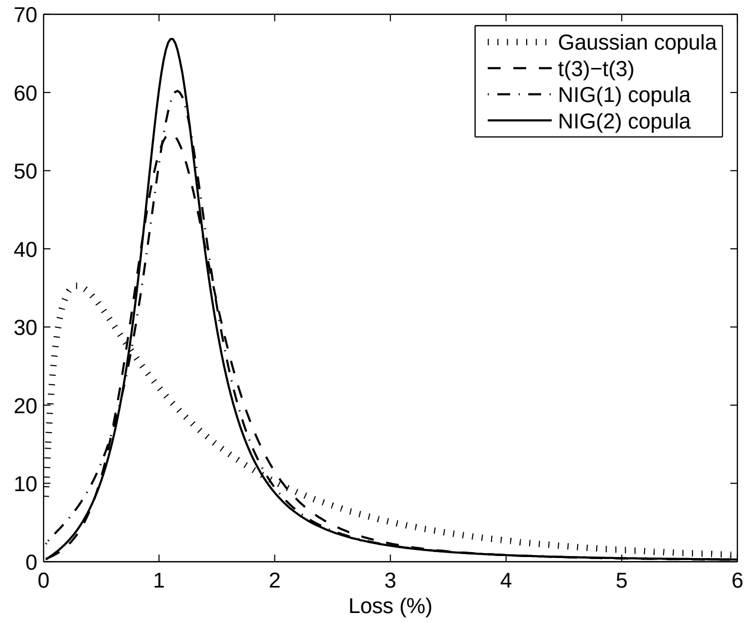
	Market	Gaussian	t(4)-t(4)	t(3)- t(3)	NIG(1)	NIG(2)
0-3%	23.53%	23.53%	23.53%	23.53%	23.53%	23.53%
3-6%	62.75 bp	140.46 bp	73.3 bp	53.88 bp	62.75 bp	62.75 bp
6-9%	18 bp	29.91 bp	28.01 bp	23.94 bp	27.9 bp	27.76 bp
9-12%	9.25 bp	7.41 bp	16.53 bp	15.96 bp	17.64 bp	17.42 bp
12-22%	3.75 bp	0.8 bp	8.68 bp	9.94 bp	9.79 bp	9.6 bp
absolute error		94.41 bp	32.82 bp	27.82 bp	24.34 bp	23.77 bp
correlation		15.72%	19.83%	18.81%	16.21%	15.94%
α					0.4794	0.6020
β					0	-0.1605
comp. time		0.5 s	12.6 s	11 s	1.5 s	1.6 s

Exhibit 1: Pricing DJ iTraxx tranches with the LHP model based on different distributions



(a) Cumulative distribution function



(b) Density function

Exhibit 2: Portfolio loss distribution from LHP model

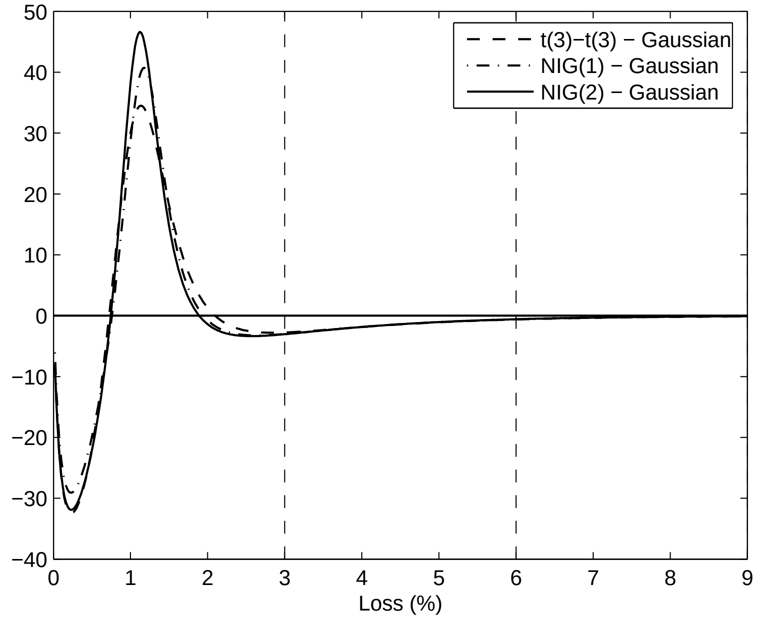


Exhibit 3: Difference between modified and Gaussian loss density

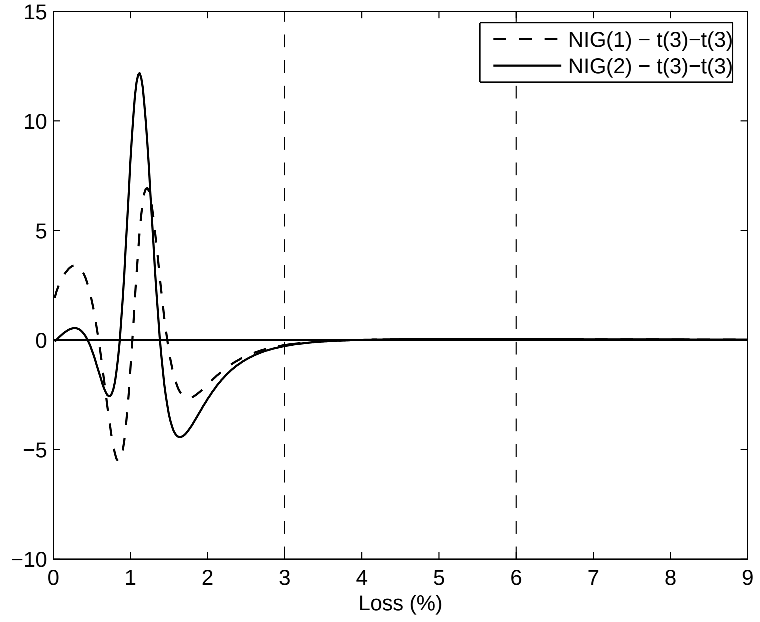


Exhibit 4: Difference between double t and NIG loss density

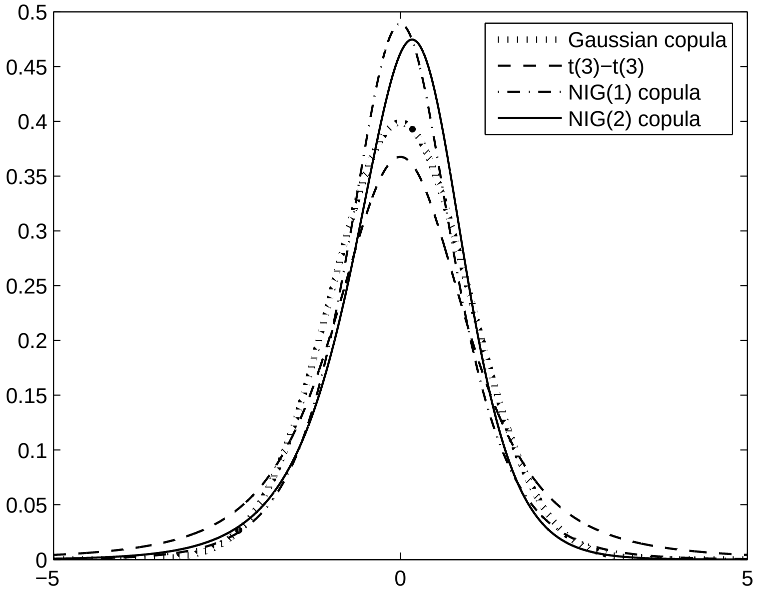


Exhibit 5: Distribution of asset returns

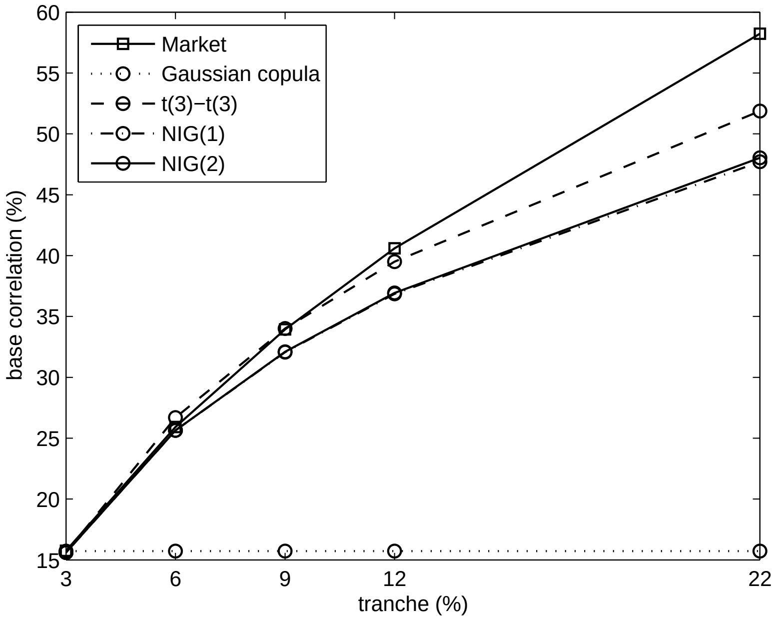
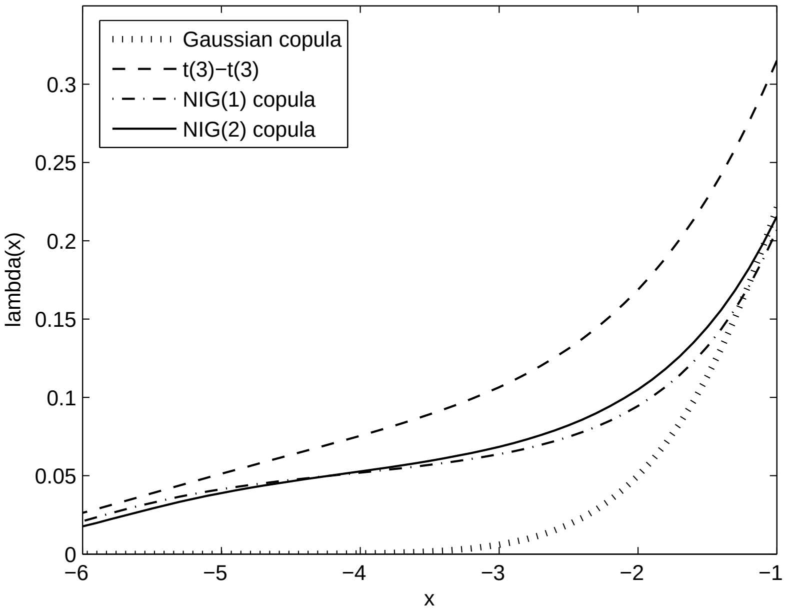
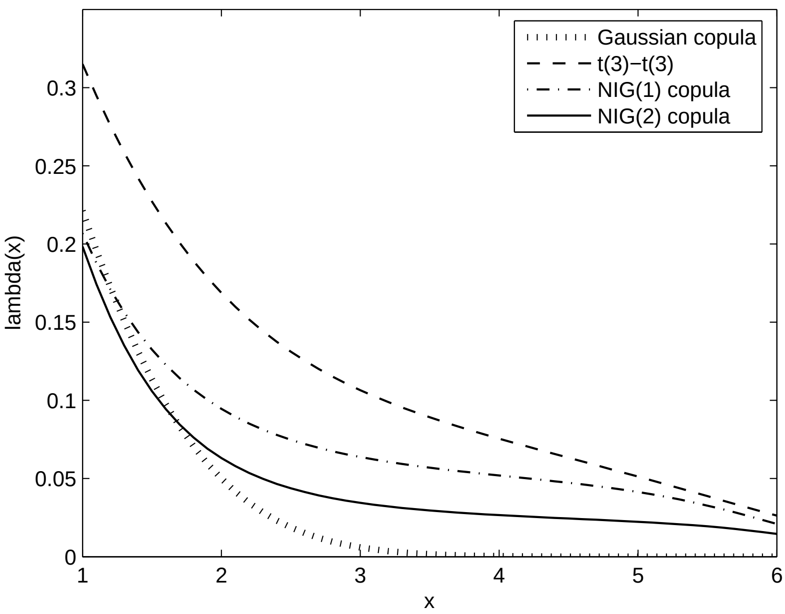


Exhibit 6: Implied base correlation



(a) Lower tail dependence



(b) Upper tail dependence

Exhibit 7: Tail dependence coefficients

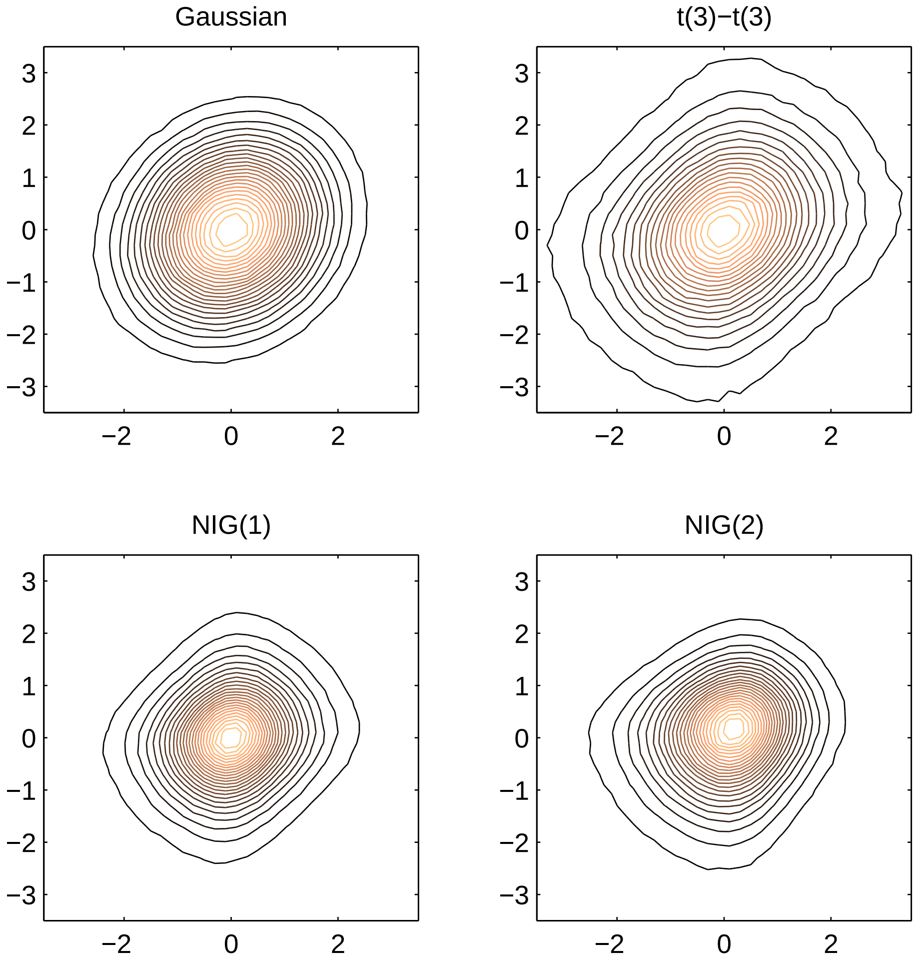


Exhibit 8: Density contours of one factor copulas