Pricing of a CDO on stochastically correlated underlyings

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Abstract

In this paper, we propose a method to price Collateralized debt obligations (CDO) within Merton's structural model on underlyings with a stochastic mean-reverting covariance dependence. There are two key elements in our development, first we reduce dimensionality and complexity using principal component analysis on the assets' covariance matrix. Second, we approximate this continuous multidimensional structure using a tree method. Trinomial-tree models can be developed for both the principal components and the eigenvalues assuming the eigenvectors constant over time and the eigenvalues stochastic. Our method allows us to compute the joint default probabilities for k defaults of stochastically correlated underlyings and the value of CDOs, without having lost much accuracy. Furthermore we provide a method to estimate the parameters to fit the model.

Key words: stochastic covariance matrix, CDO, trinomial-trees, principal component analysis

JEL classification: G13, C63

1 Introduction

The British Banking Association estimates in its Credit Derivatives Report 2006 that by the end of 2006 the size of the global credit derivatives market will be $ 20 trillion. Two of the most popular examples of credit derivatives are credit default swaps (CDS) and collateralised debt obligations (CDO). These credit derivative products strongly depend on the joint behavior of the underlying companies, i.e. the covariances. These covariances seem to change stochastically. The recent popularity of discrete stochastic correlation models like the CGARCH proposed by Engle (2002) supports this idea. On the other hand, some of the most popular procedures to price credit derivatives are non-stochastic in their correlation structures, e.g. the Linear Factor Model (LFM) (see Hull and White (2004)), the Factor-Copula Model (FCM) (see Laurent and Gregory (2003)) or the Intensity-Factor Models (see Duffie and Garleanu (2001), Hurd and Kuznetsov (2005)). There is, nevertheless, one recent attempt, the Structural Model (see Hull and White (2005)), where this feature is added within a factor model framework.

The dependence structure implied by these models and the simplicity of the pricing scheme are the key elements for a good pricing performance. In this chapter, we suggest a method which allows us to value CDOs on assets with a stochastic mean-reverting covariance matrix. To reduce dimensionality and complexity we apply Principal Component Analysis on the assets' instantaneous covariance matrix. Other publications applying principal component analysis are in support of the fact that two to three eigenvalues are sufficient to describe most of the variation in the portfolio (see Alexander (2000), (2001)). Thus, we use the first two eigenvalues and eigenvectors to compute the principal components of the Brownian motions of the assets. For these eigenvalues and principal components we suggest a system of stochastic processes and develop a trinomial-tree model to approximate the system. In this model the respective trees for the principal components and their volatilities, the eigenvalues, can be combined easily as we assume that their processes are independent. Thus, the joint default probabilities can be found by simply multiplying the marginal probabilities. We provide a path-independent method to approximate the means of the singular assets, which allows us to compute company values based on the values of the principal components and their respective probability. Using these asset values at maturity, the scenarios in which companies default can be calculated. As their probability is known the values of the tranches of the CDO can be computed. Furthermore, we provide a method to fit our model to market data.

In Section 2 the model is outlined in the following sequence, relation between prices and principal components, the actual modelling of the components and then the tree approximation. We derive a trinomial-tree model for the principal components in Section 3 and for the eigenvalues in Section 4. The combined tree system is described in Section 5. Section 6 introduces the parameter estimation method and Section 7 prices a CDO for an example. We conclude in Section 8.

2 Outline of the Model

We consider m different companies and define $V_i(t)$ to be the value of the assets of a company $i,\ i \in \{1, \ldots, m\}$ at time t. The system of processes is defined on a filtered probability space $(\Omega, \mathcal{F}, \tilde{\mathcal{Q}}, \mathbb{F})$ where $\mathcal{F}_0$ contains all subsets of the $(\tilde{\mathcal{Q}}-)$ null sets of $\mathcal{F}$ and $\mathbb{F}$ is right-continuous. As we assume the market to be arbitrage-free the processes are defined under the risk neutral measure $\tilde{\mathcal{Q}}$. The dynamics of V_i are assumed to follow a Geometric Brownian motion

$$dV_i = rV_i dt + \sigma_i V_i dW_i,\ i \in \{1, \ldots, m\},$$

where r is the constant risk-free rate of return and σ_i the volatility parameter, and W_i is a Brownian motion. Each company is assumed to be funded by equity S_i as well as one bond with face value K_i and present value L_i. The approach of Merton (1974) uses V_i to pay off debt at maturity of the contract. If V_i is insufficient to repay the debt, the company defaults. Thus, at maturity both, equity S_i and the bond L_i can be viewed as derivatives on the value V_i of the firm's assets, and the payoffs at maturity time T can be described as follows:

$$\begin{aligned} L_i(V_i, T) &= \min(K_i, V_i(T)) \\ S_i(V_i, T) &= \max(V_i(T) - K_i, 0) \end{aligned}$$

The payoff of the bond at maturity time T is the minimum of the face value K_i of the bond and the firm value $V_i(T)$, i.e. when the company value $V_i(T)$ falls below the face value K_i, the default of the company i is triggered. In this case the value of the bond L_i is equal to V_i (see Merton 1974). The payoff of the shares is identical to the payoff of a European call option on the firm value with strike price K_i. As the companies' values are assumed to follow a log normal distribution, $V_i(t) = V_i(0)e^{(r-\frac{\sigma_i^2}{2})t+\tilde{W}_i(t)}$, where $\tilde{W}_i(t) = \sigma_i W_i(t)$ is a Brownian motion, i.e. normally distributed with expectation 0 and variance $\sigma_i^2 t$. In our framework we assume that the covariance matrix follows a mean-reverting process. To handle this complexity and to reduce dimensionality implied by the m companies in the portfolio we will apply Principal Component Analysis on $\tilde{W} = (\tilde{W}_1, \ldots, \tilde{W}_m)$.

2.1 Reduction of the complexity: Principal Component Analysis

Principal Component Analysis (see Alexander (2000), (2001) and Joliffe (2002)) will allow to find m uncorrelated variables at each time t, called the principal components of $\tilde{W}(t)$. Each principal component is a simple linear combination of the original returns as we can see below. Moreover, it is possible to state how much of the original variation in the data is explained by each principal component, which are ordered according to the amount of variation they explain. Now let the stochastic covariance matrix $\Sigma(t)$ of $\tilde{W}(t)$ be

$$\Sigma(t) = AD(t)A', \tag{1}$$

where A is a $(m \times m)$ orthogonal (time independent) matrix with columns represented by the eigenvectors of $\Sigma(t)$ and $D(t)$ is a (time dependent) $(m \times m)$ diagonal matrix with the respective eigenvalues on its diagonal, which are stochastic, i.e. $D(t) = diag(\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t))$. The principal components of $\tilde{W}$ are given by

$$B = \tilde{W}A \tag{2}$$

Hence, a linear transformation of the original risk factor returns has been made in such a way that transformed risk factors are orthogonal, i.e. they have zero instantaneous correlation. The new risk factors are ordered by the amount of the stationary variation they explain. As A is a orthogonal matrix the relationship (2) is equivalent to $\tilde{W} = BA'$, i.e. $\tilde{W}_i(t) = a_{i1}B_1(t) + a_{i2}B_2(t) + \ldots + a_{im}B_m(t)$. Most of the time the first 2-3 eigenvectors are sufficient to describe more than 90% of the variation in the system (see Alexander (2000), (2001)), which allows us to reduce the complexity and dimensionality of our system by removing the eigenvectors associated to low stationary volatility. Thus, we set:

$$B^*(t) = \tilde{W}(t)A^*, \; B^* = (B_1(t), \ldots, B_{m_1}(t)) \tag{3}$$

where A^* includes the first m_1 columns of A, i.e. it is a $m \times m_1$ orthogonal matrix. Therefore, the $B_k(t)$, the kth column of $B(t)$, have zero correlation and $B_k(t) \sim N(0, D(t))$. As we assume the covariance matrix of the companies in the portfolio to be stochastic, the covariance matrix of $B^*(t)$, $D(t) = diag(\lambda_1(t), \lambda_2(t))$, is also stochastic because it is the result of a linear transformation. In terms of the original data $\Sigma(t)$, $\tilde{W}_i(t)$ and $V_i(t)$, the

transformation (3) is equivalent to

$$\begin{aligned}
\Sigma(t) &= A^* D(t) A^{*'} + \Sigma_\epsilon(t) && (4)\\
\tilde{W}_i(t) &= a_{i1} B_1(t) + \ldots + a_{im_1} B_{m_1}(t) + \epsilon_i && (5)\\
\ln V_i(t) &= \mu_i(t) + \tilde{W}_i(t) + \ln V_i(0), && (6)
\end{aligned}$$

where $\mu_i(t)$, the mean of the log returns of the company i, $i \in \{1, \ldots, m\}$ in t, is a function of the volatilities of the singular time steps. ϵ_i describes the noise caused by only including two eigenvectors and Σ_ϵ its variation. In the following we will ignore ϵ_i and therefore Σ_ϵ, the rationale for this is based on the observable fact that the remaining components have low stationary volatility and therefore they, in average, explain poorly the instantaneous volatility. The process for $B(t)$ will be described next.

2.2 Underlying Processes

We suggest the following model for the underlying system of principal components and eigenvectors k, j, where $k, j \in \{1, ..., m_!\}$:

$$\begin{aligned}
dB_k &= \sqrt{\lambda_k} dZ_k && (7)\\
d\lambda_k &= d_k(b_k - \lambda_k)dt + c_k\sqrt{\lambda_k} dQ_k && (8)\\
E(dZ_k dQ_k) &= 0 && (9)\\
E(dZ_k dZ_j) &= 0 \quad \forall\ k \neq j && (10)\\
E(dQ_k dQ_j) &= 0, && (11)
\end{aligned}$$

here d_k, b_k and c_k are fixed constants, λ_k is the kth eigenvalue and dZ_k and dQ_k are independent Wiener processes[1]. These processes are presented under the risk neutral measure, therefore the parameter b_k has been already corrected to account for the change of measure.
This is a stochastic volatility model for the principal components. The components' eigenvalues, which represent their volatility, follow the popular Cox-Ingersoll-Ross model (also known by the work of Heston 1993).

Remark 1. *Some general clarifications:*

[1]The processes for $B_k(t)$ and $d\lambda_k$ are known to exist. The original data $\tilde{W}_i(t)$, $S_i(t)$, $i \in \{1, \ldots, m\}$ and $\Sigma(t)$ can be expressed as a linear transformation of $B_k(t)$, $D_k(t)$ and A (Equations 4-6). Therefore the processes for W_i, V_i and $\Sigma(t)$ exist.

1. *Formulas 1 to 11 lead to a multidimensional stochastic volatility model for the stock prices.*

2. *From a theoretical viewpoint, if $m_1 < \frac{m}{2}$ then this system could lead to arbitrage opportunities. Nevertheless, from a practical side, those opportunities should be insignificant as long as the error ϵ_i is small.*

3. *For estimation purposes, in Section 6, we will assume that the process for the eigenvalue is the same under both, the historical and the risk neutral measure.*

2.3 Tree Building

Starting in $t = 0$ with

$$B^*(0) = \tilde{W}(0)A^*, \tag{12}$$

we are going to build a combined tree model for the stochastic volatility and the underlying B^* to get values and probabilities for B^* and D at maturity T. As seen above, these results can be expressed in terms of the original variables $\tilde{W}_i(T)$ and $V_i(T),\ i \in \{1, \ldots, m\}$:

$$\begin{aligned} \Sigma_T &= A^* D(T) A^{*'} && (13)\\ \tilde{W}_i(T) &= a_{i1}B_1(T) + \ldots + a_{i2}B_2(T) + \epsilon_i && (14)\\ \ln V_i(T) &= \mu_i(T) + \tilde{W}_i(T) + \ln V_i(0), && (15) \end{aligned}$$

These data allow us to calculate how many companies default, assuming we know the face values K_i of the bonds and the respective probabilities, i.e. all the information needed to compute the value of the tranches of a CDO, without having lost much accuracy.

3 Trinomial-Tree Approximation for the Transform of the Brownian motion

For simplicity of explanation, we set $m_1 = 2$ from this section on. The process (7) for $dB_k,\ k \in \{1, 2\}$, is approximated using a trinomial-tree because in this framework the spacing parameter can be set independent from the volatility of the process. In the following, the nodes are denoted by (j, t), where j is

the number of upwards movements and t indicates the number of time steps passed since $t = 0$, i.e. the root of the tree. Let $B_k(t) = B_k(0) + j\Delta B_k$, then

$$\Delta B_k(t) := B_k(t+1) - B_k(t) = \begin{cases} \Delta B_k & \text{with probability} \quad p_{j,j+1,t} \\ 0 & \text{with probability} \quad p_{j,j,t} \\ -\Delta B_k & \text{with probability} \quad p_{j,j-1,t} \end{cases} \tag{16}$$

The convergence of this discrete three-point distribution to the continuous process is ensured by matching the first two moments. Note that $E(B_k(0)) = 0$. Again, let $B_k(t) = B_k(0) + j\Delta B_k = j\Delta B_k$. Then,

$$\begin{aligned} E(\Delta B_k(t)) = p_{j,j+1,t}\Delta B_k - p_{j,j-1,t}\Delta B_k &= 0 && (17) \\ Var(\Delta B_k(t)) = E(\Delta B_k^2(t)) &= p_{j,j+1,t}\Delta B_k^2 + p_{j,j-1,t}\Delta B_k^2 && (18) \\ &= \lambda_k(j\Delta B_k, (t-1)\Delta t)\Delta t && (19) \\ p_{j,j+1,t} + p_{j,j,t} + p_{j,j-1,t} &= 1 && (20) \end{aligned}$$

From Equation (17) follows

$$p_{j,j+1,t} = p_{j,j-1,t} \tag{21}$$

Substituting Equation (21) in Equation (19) we get

$$\Delta B_k^2(2p_{j,j+1,t}) = \lambda_k(j\Delta B_k, (t-1)\Delta t)\Delta t \tag{22}$$

Solving this for $p_{j,j+1,t}$ we obtain

$$p_{j,j+1,t} = \frac{1}{2}\frac{\lambda_k(j\Delta B_k, (t-1)\Delta t)}{\Delta B_k^2}\Delta t \tag{23}$$

and

$$\begin{aligned} p_{j,j-1,t} &= \frac{1}{2}\frac{\lambda_k(j\Delta B_k, (t-1)\Delta t)}{\Delta B_k^2}\Delta t && (24) \\ p_{j,j,t} &= 1 - \frac{\lambda_k(j\Delta B_k, (t-1)\Delta t)}{\Delta B_k^2}\Delta t, && (25) \end{aligned}$$

To guarantee positive probabilities $p_{j,j+1,t}, p_{j,j,t}$ and $p_{j,j-1,t}$ we choose appropriate spacing and timing parameters. From equations (23), (24) and (25) it can be shown that the probabilities are positive if

$$\lambda_k(j\Delta B_k, (t-1)\Delta t) < \frac{\Delta B_k^2}{\Delta t}. \tag{26}$$

is satisfied (see Escobar et al. (2007)). Increasing the number of time steps, Hull and White (1990) and Brennan and Schwartz (1978) find it desirable to keep the ratio $\frac{\Delta B_k^2}{\Delta t}$ constant to ensure convergence. In order to develop a recombining tree with a spacing parameter not depending on the volatility, i.e. the eigenvalue, the eigenvalue has to be restricted. However, in order not to restrain the model too much we use the following relationship between the variances of the original data and the eigenvalues

$$\begin{aligned}\sum_{i=1}^{m} Var(V_i(t)) = trace(\Sigma(t)) &= trace(\Sigma(t)AA') \\ &= trace(A'\Sigma(t)A) = trace(D(t)) = \sum_{k=1}^{m} \lambda_k(t)\end{aligned}$$

(see Joliffe (2002)). Thus, we set

$$\Delta B_k^2 \equiv \gamma \Delta t \sum_{k=1}^{m} \lambda_k(0) \tag{27}$$

and inequality (26) transforms to

$$\lambda_k(j\Delta B_k, (t-1)\Delta t) < \gamma \sum_{k=1}^{m} \lambda_k(0). \tag{28}$$

A sufficient choice of γ is possible as all $\lambda_k(t)$ are bounded in the trinomial tree. The tree building and the combination of the trees are shown for the first two principal components. Table (1) illustrates the possible movements of B_1 and B_2 after two time steps. The probabilities of each joint combination are obtained by simply multiplying the probabilities of the marginals as the marginals are uncorrelated. These marginal probabilities are, however, influenced by the volatilities $\sqrt{\lambda_1(t-1)}$ and $\sqrt{\lambda_2(t-1)}$ respectively.

4 Trinomial-Tree for the Eigenvalue Matrix

The process (8) is implemented using a trinomial-tree suggested by Hull and White (1990). This method requires a constant volatility parameter for

Table 1: Possible Movements of $B_1(t)$ and $B_2(t)$ after two time steps

$2\Delta B_1(t)$ $2\Delta B_2(t)$	$2\Delta B_1(t)$ $\Delta B_2(t)$	$2\Delta B_1(t)$ 0	$2\Delta B_1(t)$ $-\Delta B_2(t)$	$2\Delta B_1(t)$ $-2\Delta B_2(t)$
$\Delta B_1(t)$ $2\Delta B_2(t)$	$\Delta B_1(t)$ $\Delta B_2(t)$	$\Delta B_1(t)$ 0	$\Delta B_1(t)$ $-\Delta B_2(t)$	$\Delta B_1(t)$ $-2\Delta B_2(t)$
0 $2\Delta B_2(t)$	0 $\Delta B_2(t)$	0 0	0 $-\Delta B_2(t)$	0 $-2\Delta B_2(t)$
$-\Delta B_1(t)$ $2\Delta B_2(t)$	$-\Delta B_1(t)$ $\Delta B_2(t)$	$-\Delta B_1(t)$ 0	$-\Delta B_1(t)$ $-\Delta B_2(t)$	$-\Delta B_1(t)$ $-2\Delta B_2(t)$
$-2\Delta B_1(t)$ $2\Delta B_2(t)$	$-2\Delta B_1(t)$ $\Delta B_2(t)$	$-2\Delta B_1(t)$ 0	$-2\Delta B_1(t)$ $-\Delta B_2(t)$	$-2\Delta B_1(t)$ $-2\Delta B_2(t)$

the process, which can be obtained with the following transformation of the process:

$$\phi_k = \sqrt{\lambda_k},\ k \in \{1, 2\} \tag{29}$$

Using Itô's Lemma, this leads to

$$\begin{aligned} d\phi_k &= d\sqrt{\lambda_k} = (\frac{1}{2\sqrt{\lambda_k}}(b_k d_k - d_k\lambda_k) - \frac{1}{8}\lambda_k^{-\frac{3}{2}}c_k^2\lambda_k)dt + \frac{c_k}{2}dQ_k \\ &= (\frac{b_k d_k}{2\phi_k} - \frac{d_k}{2}\phi_k - \frac{1}{8}c_k^2\frac{1}{\phi_k})dt + \frac{c_k}{2}dQ_k \\ &= (\frac{4b_k d_k - c_k^2}{8}\frac{1}{\phi_k} - \frac{d_k}{2}\phi_k)dt + \frac{c_k}{2}dQ_k \\ &= q_k dt + \nu_k dQ_k, \end{aligned}$$

where $\nu_k = \frac{c_k}{2}$, $q_k = \frac{\alpha_{1,k}}{\phi_k} - \alpha_{2,k}\phi_k$ and $\alpha_{1,k} = \frac{4d_k b_k - c_k^2}{8}, \alpha_{2,k} = \frac{d_k}{2}$.
Again the lifetime of the derivative is divided in $n = \frac{T}{\Delta t}$ equal time steps, where Δt is the length of one such time step. After each time step the tree branches out. The nodes are denoted by (l, t), where l is the number of upwards movements of the eigenvalue, i.e. the value $\phi_k(l, t) = \phi_k(0) + l\Delta\lambda$, and t indicates the number of time steps passed since $t = 0$. As before, we assume that λ_k can increase, move aside, or decrease. However, as q is unbounded, the model we used before may not converge. Thus, this method was modified by Hull and White (1990). Beside the possibility that $\Delta\phi_k$ is equal to $\Delta\phi_k(l, t), 0$ and $-\Delta\phi_k(l, t)$ two other branching methods are relevant for this implementation. The three methods are illustrated in Figures 1-3, where κ is the move increment.

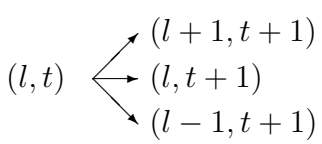

Figure 1: Relation between time step t and $t+1$ when $\kappa = l$

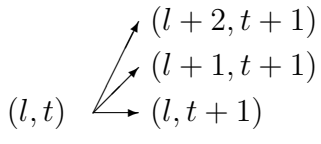

Figure 2: Relation between time step t and $t+1$ when $\kappa = l+1$

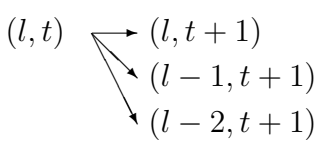

Figure 3: Relation between time step t and $t+1$ when $\kappa = l-1$

To ensure convergence the probabilities are set to (see Escobar et al. (2007)):

$$p_{l,\kappa+1} = \frac{\nu_k^2 \Delta t}{2(\Delta\phi_k)^2} + \frac{\eta_k^2}{2(\Delta\phi_k)^2} + \frac{\eta_k}{2\Delta\phi_k} \tag{30}$$

$$p_{l,\kappa} = 1 - \frac{\nu_k^2 \Delta t}{(\Delta\phi_k)^2} - \frac{\eta_k^2}{(\Delta\phi_k)^2} \tag{31}$$

$$p_{l,\kappa-1} = \frac{\nu_k^2 \Delta t}{2(\Delta\phi_k)^2} + \frac{\eta_k^2}{2(\Delta\phi_k)^2} - \frac{\eta_k}{2\Delta\phi_k}, \tag{32}$$

where

$$\eta = q_k(l,t)\Delta t + (l-\kappa)\Delta\phi_k$$

When $\Delta\phi_k$ is set to $\nu_k\sqrt{3\Delta t}$ the following dynamic rules for the choice of κ can be implemented to ensure positive probabilities:

$$\kappa = \begin{cases} l+1 & \text{if} \quad \frac{q_k(l,t)\Delta t}{\Delta\phi} \geq \sqrt{\frac{2}{3}} \\ l & \text{if} \quad -\sqrt{\frac{2}{3}} < \frac{q_k(l,t)\Delta t}{\Delta\phi} < \sqrt{\frac{2}{3}} \\ l-1 & \text{if} \quad \frac{q_k(l,t)\Delta t}{\Delta\phi} \leq -\sqrt{\frac{2}{3}} \end{cases} \tag{33}$$

These dynamic rules of choice for κ imply minimum and maximum values for $\phi_k(t)$:

$$-\sqrt{\tfrac{2}{3}} \leq \tfrac{q_k(l,t)\Delta t}{\Delta\phi} \leq \sqrt{\tfrac{2}{3}} \;\Leftrightarrow\; \phi_{k,min} = \tfrac{-\beta+\sqrt{\beta^2+4\alpha_1\alpha_2}}{2\alpha_2} \leq \phi_k(t) \leq \tfrac{\beta+\sqrt{\beta^2+4\alpha_1\alpha_2}}{2\alpha_2} = \phi_{k,max} \tag{34}$$

where $\beta = \sqrt{\frac{2}{3}}\frac{\Delta\phi_k}{\Delta t}, \alpha_1 = \frac{4db-c^2}{8}, \alpha_2 = \frac{1}{2}d$.
The branching method is changed to $\kappa = l-1$ at a node (ψ, t), where ψ is the largest integer that $\phi_k(\psi,t) = \phi_k(0)+\psi\Delta\phi \leq \phi_{k,max}$ and to $\kappa = l+1$ at a node (ζ, t), where ζ is the smallest integer that $\phi_k(\zeta,t) = \phi_k(0) + \zeta\Delta\phi \geq \phi_{k,min}$.

5 Combined Tree for Principal Components and Eigenvalues

In order to obtain spacing parameters, which are independent from the volatility in the trees for the principal components, we had to restrict the volatility (see (28)) generated by the trees for $\sqrt{\lambda_k},\ k \in \{1,2\}$. Thus, additionally to the restrictions on the volatility imposed by (34) it has to hold that

$$\phi_k(j\Delta B_k,(t-1)\Delta t) < \sqrt{\gamma \sum_{k=1}^{m} \lambda_k(0)} \tag{35}$$

Hence, whichever of the restrictions (34) and (35) is reached earlier in a time step t it causes the tree for ϕ_k to change the branching method in this node (s,t). s is $\min(\psi,\vartheta)$, where ϑ is the largest integer that $\phi_k(\vartheta,t) = \phi_k(0) + \vartheta\Delta\phi \leq \sqrt{\gamma \sum_{k=1}^{m} \lambda_k(0)}$ in (35) and ψ is the largest integer that $\phi_k(\psi,t) = \phi_k(0) + \psi\Delta\phi \leq \phi_{k,max}$ in (34). As we assume the Brownian motions of the principal components of the assets and of the eigenvalues to be independent, the joint probabilities can be obtained by simply multiplying the marginal probabilities. The probabilities for the movements of the principal components ((23) to (25)) also apply in the case of stochastic volatilities. The nodes in the combined tree are denoted by (j,w,l,v,t), where j and w indicate the number of up or down moves of the first and the second principal component respectively. l and v specify the level of the two eigenvalues. The trees of the eigenvalues and the principal components are arranged in series, i.e. the eigenvalues in t influence the probabilities of the principal components to move up or down in $t+1$ (in the tree the principal components increase or decrease n times, the eigenvalues $n-1$ times until T). A particular node in the combined tree branches in 3^4 different nodes in the next time step.

5.1 Computation of the Means

In T, $\mu_i(T),\ i \in \{1,\ldots,m\}$, has to be added to the retransformed Brownian motions $\tilde{W}_i(T)$. $\mu_i(T)$ is actually the sum of the means of the singular time steps and can be obtained by combining the two trees of the eigenvalues to a tree for the variances of the underlyings with nodes (l,v,t) to $(3)^{(n-1)}$ nodes

in T, assuming n time steps:

$$\mu_i(l,v,t|l,v,t-1;\ldots;l,v,t=0) = \sum_{t=1}^{n}(r - \frac{\sigma_i(l,v,t)^2}{2})\Delta t, \quad \forall \quad \text{combinations} \quad l,v,$$

where $\sigma_i(l,v,t)^2$ is computed by

$$\lambda_1(l,t)a_{1i}^{*2} + \lambda_2(v,t)a_{2i}^{*2} \quad \forall \quad \text{combinations} \quad l,v. \tag{36}$$

However, the determination of the exact means would result in a path-dependent pricing as $\mu_i(T)$ depends on the path in the combined volatility tree. This results in the computation of $3^{2\cdot(n-1)}$ different paths, where n is the number of time steps computed in the tree. The distribution of the values of the means in T is illustrated for an example in section 7.
We will compute the mean at each node (l,v,t) in the variance tree by weighting the means of the predecessing nodes with the conditional probabilities to reach the respective node (l,v,t). In this way each node of the combined variance tree in time step n gives rise to one mean value $\mu_i(T)$, i.e. there are $((n-1)\cdot 2+1)^2$ different mean values in T. This is illustrated in Figure 4 for $n=2$. In Section 7 we compare the distribution of the values resulting from this method with the path-dependent computation method. This comparison shows that the distributions are similar.
These mean values allow us to express the principal components and eigenvalues in T in terms of $\ln V_i$ and V_i, $i \in \{1,\ldots,m\}$:

$$\Sigma(T) \approx A^* D(T) A^{*'} \tag{37}$$

$$\ln V_i(T) \approx \ln V_i(0) + \mu_i(T) + \tilde{W}_i(T), \tag{38}$$

$$V_i(T) \approx V_i(0)\exp\left(\mu_i(T) + \tilde{W}_i(T)\right) \tag{39}$$

By subtracting the face values D_i from V_i it can be determined if and how many companies default in each scenario. The pay-off function of each tranche of the CDO can then be weighted with the joint probabilities of the respective scenario (i,j,l,v,T) and discounted, which results in the value of the tranche in $t=0$.

6 Parameter Estimation

The estimation procedure we propose to calibrate the parameters of our model is inspired by the work of Genon-Catalot et al. (2000) for stochastic

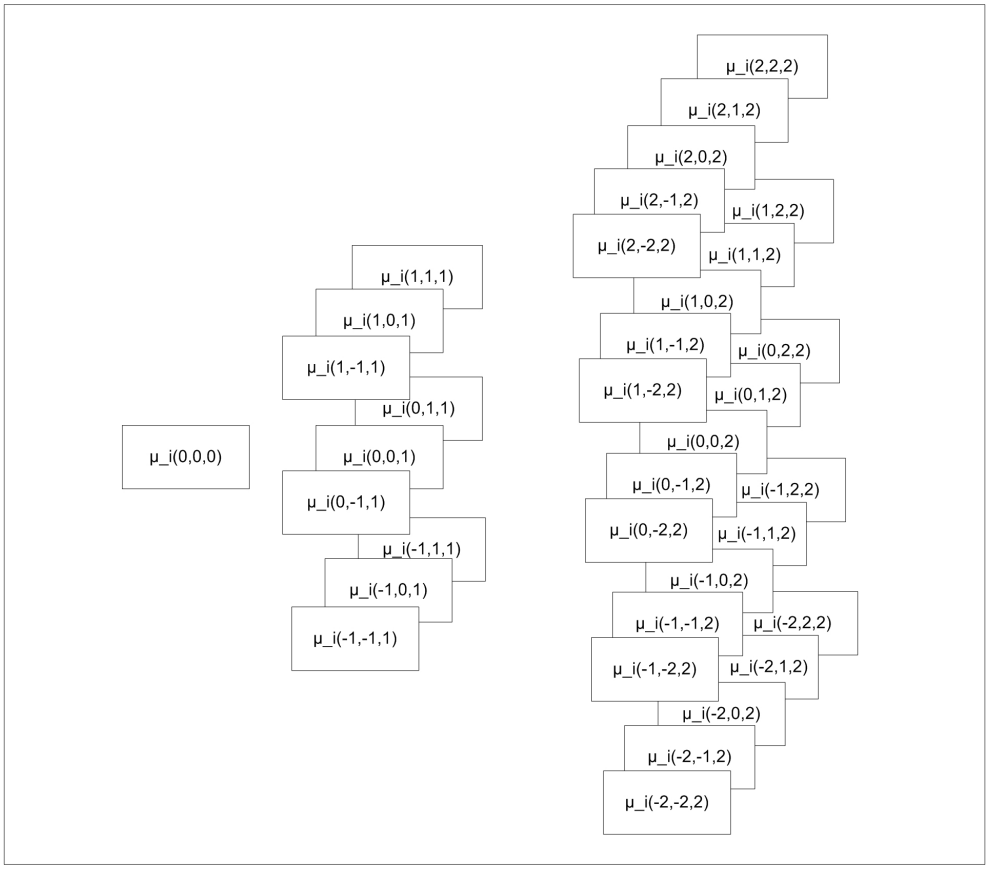

Figure 4: Tree for the Mean Values for $n = 2$ and two eigenvalues

volatility models in the case of a fixed sampling interval Δt. Our method is based on the idea of multidimensional Hidden Markov models.
It is important to realize that the data could come from two sources either companies with observable asset values, for which the estimation will be applied directly, or companies with only observable equity prices. For the second case, we relies on the assumption that the volatility structure of assets and equity is the same. This holds true, for example, in the case of companies with a relatively small debt (leverage).
In this section, we will first explain the relation between Hidden Markov Models and stochastic volatility (section 7.1), then this concepts will be applied for our multidimensional stochastic volatility model (section 7.2).

6.1 Stochastic volatility models as hidden Markov models

Genon-Catalot et al. (2000) show that a stochastic volatility model can be viewed as a hidden Markov model.

Definition 1. *(Hidden Markov model)(Genon-Catalot et al. (2000) Definition 3.1)*
A stochastic process $(S(i), i \geq 1)$, with state space $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$, is a hidden Markov model if the following hold:

- *(Hidden chain) We are given (but do not observe) a strictly stationary Markov chain $U(1), U(2), \ldots, U(z), \ldots$ with state space $(\mathcal{U}, \mathcal{B}(\mathcal{U}))$.*
- *For all z, given $(U(1), U(2), \ldots, U(z))$, the $S(i)$, $i = 1, \ldots, z$, are conditionally independent, and the conditional distribution of $S(i)$ depends only on $U(i)$.*
- *The conditional distribution of $S(i)$ given $U(i) = u$ does not depend on i.*

Proposition 1. *(Genon et al. (2000) Proposition 3.1)*
The process $(S(i), i \geq 1)$ is strictly stationary. If the hidden Markov chain $(U(i), i \geq 1)$ is ergodic, then $(S(i), i \geq 1)$ is also ergodic.

For a proof see Genon et al. (2000).
Now consider a Markov process which is defined by a stochastic equation

$$dX = \mu(X)dt + \sigma(X)dZ, \quad X_0 = \eta, \tag{40}$$

where W is a standard Brownian motion and η is a real random variable defined on Ω and independent of Z. Genon et al. (2000) make the standard assumptions on functions $b(x)$ and $a(x)$, ensuring that the solution (see Assumption 1) of (40) is a positive recurrent diffusion on an interval (l, r), $-\infty \leq l < r \leq \infty$, (see Assumption 2) and a strictly stationary ergodic time-reversible process (see Assumption 3).

Assumption 1.
The functions $\mu(x)$ and $\sigma(x)$ are defined on (l,r), and satisfy

$$\mu(x) \in C^1(l,r), \sigma^2(x) \in C^2(l,r), \sigma(x) > 0 \ \forall x \in (l,r), \tag{41}$$

and

$$\exists K > 0, \forall x \in (l,r), |\mu(x)| \leq K(1+|x|) \text{ and } \sigma^2(x) \leq K(1+x^2). \tag{42}$$

For $x_0 \in (l,r)$, define the scale and speed densities of diffusion (X_t),

$$s(x) = \exp(-2\int_{x_0}^{x} \frac{\mu(u)}{\sigma^2(u)}du), \quad m(x) = \frac{1}{\sigma^2(x)s(x)}. \tag{43}$$

Assumption 2.

$$\int_l^{\infty} s(x)dx = \infty, \int_{-\infty}^{r} s(x)dx = \infty, \int_l^r m(x)dx = M < \infty. \tag{44}$$

Let us define the stationary density

$$\pi^*(x) = \frac{1}{M} m(x) 1_{x\in(l,r)}. \tag{45}$$

Assumption 3.
The initial random variable η has distribution $\pi^*(dx) = \pi^*(x)dx$.

Now let $(Y,V)_{t\geq 0}$ be a two-dimensional diffusion process given by

$$dY = \sigma dZ, Y(0) = 0, \tag{46}$$

$$V = \sigma^2, dV = d(b-V)dt + c\sqrt{V}dQ, V(0) = \eta, \tag{47}$$

where η is a random variable, independent of (Z,Q), and assume that Assumptions 1 to 3 hold for $(Y,V)_{t\geq 0}$. Then, the diffusion V is strictly stationary, ergodic and time-reversible (see Proposition 1 and Genon et al. (2000)). For positive Δt Genon-Catalot et al. (2000) define, for $i \geq 1$,

$$S(i) = \frac{1}{\sqrt{\Delta t}} \int_{(i-1)\Delta t}^{i\Delta t} \sigma_s dZ_s \tag{48}$$

and

$$U(i) = (\overline{V}(i), V(\Delta t)), \quad \overline{V}(i) = \frac{1}{\Delta t} \int_{(i-1)\Delta t}^{i\Delta t} V_s ds, \tag{49}$$

where $V(\Delta t)$ are values of V at fixed intervals Δt.

Proposition 2. *(Genon-Catalot et al. (2000) Proposition 4.1)*
Assume that the above hidden diffusion V satisfies the Assumptions 1 to 3 and that $E(V(0)^2)$ is finite. Then, $E(\overline{V}(1)) = E(\overline{V}(0)) = \beta$, and

$$E(\overline{V}(1)^2) = b^2 + \frac{c^2 b}{2d} \frac{2(d\Delta t - 1 + e^{-d\Delta t})}{d^2(\Delta t)^2}, \tag{50}$$

$$E(\overline{V}(1)\overline{V}(2)) = b^2 + \frac{c^2}{2d} b \frac{(1 - e^{-d\Delta t})^2}{d^2(\Delta t)^2}. \tag{51}$$

For a proof see Genon-Catalot et al. (2000).
The following functions of the observations

$$\widehat{m}_1 = \tfrac{1}{n} \sum_{i=1}^{n} Z(i)^2, \quad \widehat{m}_2 = \tfrac{1}{3n} \sum_{i=1}^{n} Z(i)^4, \quad \widehat{m}_{12} = \tfrac{1}{n} \sum_{i=1}^{n-1} Z(i)^2 Z(i+1)^2, \tag{52}$$

where n is the number of observations, are consistent estimators of b^2, $E(\overline{V}(1)^2)$ and $E(\overline{V}(1)\overline{V}(2))$ respectively. For a proof see Genon-Catalot et al. (2000).

6.2 Application for the parameter estimation of the eigenvalue processes

In our multidimensional model, an orthogonal transformation of the returns leads to a set of univariate independent stochastic volatility processes. Therefore we study the idea of extending univariate estimations methods as the one explain before to get our process.
For our estimation, we propose a two steps procedure. First, the constant orthogonal matrix A^* is found by singular decomposition on the stationary covariance structure. Second, we use the procedure of Genon-Catalot et al. (2000) outlined in the previous section to estimate the parameters in each principal component (SV process).

In a separate paper, the details on how the stationarity and ergodicity of the principal components implied the same attributes on the observable increments of the multidimensional log-prices will be provided. These features are necessary for our estimators to be consistent.
In the first step, we consider a diffusion vector process $V(t)$, which is observable at n discrete times with regular sampling interval Δt. Let us denote the vector of observations by $V(\Delta t)$, the vector of returns would be $R(\Delta t) = \frac{V(\Delta t)-V(\Delta(t-1))}{V(\Delta(t-1))}$. The covariance matrix of $R(\Delta t)$ is found using standard methods, Σ_1. A^* is taken as the orthogonal matrix in a singular decomposition of Σ_1.
Second step, we consider a diffusion process $(B(t), D(t))$, where we regard $B(t)$, the matrix of the principal components, as observable [2] at n discrete times with regular sampling interval Δt and denote the observations by $B(\Delta t)$. $D(t)$, the matrix of the eigenvalues, is assumed to be ergodic (see conditions on previous section) and rules the diffusion of $(B(\Delta t))$. The observations $B(t)$ can be viewed as a hidden Markov model. For Δt positive, we define, for $i \geq 1$,

$$B(i) = \frac{1}{\sqrt{\Delta t}} \int_{(i-1)\Delta t}^{i\Delta t} \sqrt{\lambda(s)} dZ \tag{53}$$

and

$$\overline{D}(i) = \frac{1}{\Delta t} \int_{(i-1)\Delta t}^{i\Delta t} D(s) ds. \tag{54}$$

We subtract in each sampling interval the mean $\overline{V}_i$ from all data series V_i to obtain m data sets whose mean is zero. For these m sets we compute the first two principal components to obtain two-dimensional $B(i)$.
Thus, for our model (7 - 11) we can apply that $E(\overline{D}(1)) = E(\overline{D}(0)) = b$, and

$$E(\overline{D}(1)^2) = b^2 + \frac{c^2}{2d} b \frac{2(d\Delta t - 1 + e^{-d\Delta t})}{d^2(\Delta t)^2}, \quad E(\overline{D}(1)\overline{D}(2)) = b^2 + \frac{c^2}{2d} b \frac{(1-e^{-d\Delta t})^2}{d^2(\Delta t)^2}. \tag{55}$$

[2] $B(t)$ is a linear transformation of the observable data. The fact that A^* is also calibrated leads to less efficient still consistent estimators.

Consistent estimators of b, $E(\overline{D}(1)^2)$ and $E(\overline{D}(1)\overline{D}(2))$ are the following functions of the observations

$$\widehat{m}_1 = \frac{1}{n}\sum_{i=1}^{n} Z(i)^2, \quad \widehat{m}_2 = \frac{1}{3n}\sum_{i=1}^{n} Z(i)^4, \quad \widehat{m}_{12} = \frac{1}{n}\sum_{i=1}^{n-1} Z(i)^2 Z(i+1)^2. \tag{56}$$

which allow us to estimate the parameters d, b, c for the diffusions of the first two eigenvalues.
The literature on estimation methods for continuous-time multidimensional stochastic volatility processes is almost non-existent, (see Broto and Ruiz 2004 for a survey). This method, like any other moment-based procedure, has several difficulties for small and medium sample sizes as well as for some ranges of the parameters (see Sørensen (2000) for a survey). We have, however, already tested the method on simulations, i.e. we tried to retrieve the parameters of the simulations using our parameter estimation method. This showed that the results for the mean-reversion level and the mean-reversion speed were quite close. On the other hand, its benefit lies in the speed of calibration, comparing to other proposed (but untested) methods like Indirect Inference, Simulated Maximum Likelihood or Markov Chain Monte Carlo, which depends on simulations, our is the fastest one.

7 Example

In the following we value the tranches of a CDO. The underlying security is a portfolio of coupon paying corporate bonds on 10 firms. We assume the face values $K_i, i \in \{1, \ldots, 10\}$ of these bonds to be 1 and assume that the process of the company values, $V(t)$, show the same characteristics as the respective observable stock price processes, $S(t)$ (based on Merton's framework, this applies either to low debt companies or those for which no information on assets is available). Thus, in order to estimate the parameters of the eigenvalue process we use the stock returns of ten companies (MMM.N, ABT.N, AA.N, AXP.N, BUD.N, AVP.N, BAX.N, BDK.N, BNI.N, BMY.N) from 1983 to October 2006. For these companies, no information regarding assets values or debt is available, only probabilities of default are known, these are used to have an idea on default thresholds for stock prices. We use the parameter estimation method described above for the mean-reversion level and the mean-reversion speed. As the method has still to be improved, considering the volatilities of the eigenvalue, we assume plausible values for the latter.

Thus, we value the tranches for the following scenario:

Basic Scenario: $r = 0.05$, Maturity = 1 year, $\Delta t = \frac{1}{n}$, $n = 3$, $b_1 = 0.2713032$, $b_2 = 0.126441$, $d_1 = d_2 = 0.5$, $c_1 = 0.14$, $c_2 = 0.12$, $K_i = 0.75 \cdot V_i(0)$

Basic Scenario: Tranche 1 = from 0 to 1 default, Tranche 2 = from 2 to 3 defaults, Tranche 3 = from 3 to 4 defaults, Tranche 4 = from 4 to 5 defaults, Tranche 5 = 5 to 10 defaults

We value the tranches using the path-dependent and the non path-dependent mean value computation in order to compare the distributions we get for the mean values in T and the values of the tranches. The results for the values of the tranches of these two methods are close (see Table 2).

Table 2: Values of the Tranches using the path dependent method

Tranche	path-dependent method	Non path-dependent method
1	0.1140942079	0.1149043351
2	0.0745104126	0.0745104126
3	0.0727544611	0.0727544611
4	0.0543120038	0.0543120038
5	0.0377722636	0.0376654103

Furthermore, we analyze the distribution of the means of the two methods. Using the Mann-Whitney Test we cannot reject the H_0 hypothesis that the two distributions are alike. The two distributions are shown in the appendix. and analyzed in Table 3. However, the values of the tranches differ more

Table 3: Moments of the Mean Values

	path-dependent Method	Non path-dependent Method
Mean	0.026241208	0.026125472
Variance	$4.92252E-06$	$5.82982E-06$
Skewness	-0.242396164	-0.195523145
Minimum	0.018953	0.018953
Maximum	0.031259	0.031259

when we assume higher volatilities for the process of the eigenvalues. In the

following we have calculated the value for the volatilities c_1=0.25 and c_2=0.13 (see Table 4).

Table 4: Values of the Tranches using the path-dependent method with higher volatilities

Tranche	path-dependent method	Non path-dependent method
1	0.1291563212	0.1423252046
2	0.0745361782	0.0745268594
3	0.0727739773	0.0727548047
4	0.0544242608	0.0544113424
4	0.0392558059	0.0544113424

The deviation in value of the first tranche can be explained by the fact that the differences in skewness between the distributions of these two methods become higher for higher volatilities of the eigenvalue process although the span of both distributions, i.e. minimum and maximum values, remains the same for both distributions (see Tables 3 and 5). This imprecision can be handled by using more time steps for the computation of the tranches when the processes of the eigenvalues are quite volatile. The two distributions are shown in the appendix and analyzed in Tables 5.

Table 5: Moments of the Mean Values

	path-dependent Method	Non path-dependent Method
Mean	0.026235437	0.026026984
Variance	$1.82868E-05$	$1.54905E-05$
Skewness	-0.432561011	-0.330387539
Minimum	0.013258	0.013258
Maximum	0.034434	0.034434

8 Summary and Conclusion

We have developed and implemented a tree model to price tranches of a CDO on a portfolio of stochastically correlated underlyings. Dimensionality and complexity can be reduced by using only the first few eigenvectors and eigenvalues of the underlyings instead of the actual values. We have built trees

for the principal components and the stochastic eigenvalues, which allows us to compute the underlying probabilities of default in this portfolio. This model relaxes the constant correlation assumption in the existing literature. Furthermore, we provide a method to match the parameters of the stochastic processes of the principal components as well as the eigenvalues to market data and show the parameter estimation and the valuation of the tranches for an example.

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