# A Hybrid-Form Model for the Prepayment-Risk-Neutral Valuation of Mortgage-Backed Securities

Andreas Kolbe *†  Rudi Zagst ‡

August 18, 2006

**Abstract**

In this paper we present a prepayment-risk-neutral valuation model for fixed-rate Mortgage-Backed Securities. Our model is based on intensity models as used in credit-risk modelling and extends existing models for individual mortgage contracts in a proportional hazard framework. The general economic environment is explicitly accounted for in the prepayment process by an additional factor which we fit to the quarterly GDP growth rate in the US. In our risk-neutral setting we account for both the fears of refinancing understatement and turnover overstatement which sometimes result in higher option-adjusted spreads (OAS) for premiums and discounts respectively. We apply our prepayment-risk-neutral pricing approach to a sample of generic 30yr GNMA MBS pass-throughs and find that our model successfully explains market prices.

*Keywords: prepayment, Mortgage-Backed Security, OAS, intensity, risk-neutral pricing, proportional hazard*

*HVB-Institute for Mathematical Finance, Munich University of Technology, Boltzmannstr. 3, D-85748 Garching (Munich), Germany, kolbe@ma.tum.de

†Support from Bayerische Landesbank (BayernLB), Risk Control Trading Sections, Munich and BayernLB Risk Control Americas, New York, is gratefully acknowledged

‡HVB-Institute for Mathematical Finance, Munich University of Technology, Boltzmannstr. 3, D-85748 Garching (Munich), Germany, zagst@ma.tum.de

# 1 Introduction, motivation and related research

Traditionally, the academic literature on the valuation of Mortgage-Backed Securities could be divided into two general categories: The structural, option-based approach where prepayment is related to a mortgagor's rational decision to exercise the prepayment option inherent in the mortgage contract and the econometric approach where an empirically estimated prepayment function, often within a proportional hazard framework, is used to forecast prepayment cash flows. Stanton (1995) and Schwartz and Torous (1989) are two examples of popular and frequently cited papers concerned with the option-based approach and the econometric approach respectively. Since the option-based models have not been able to explain market prices consistently so far without incorporating "unrealistic" assumptions about, e.g., transaction costs (mainly due to non-optimal behaviour of mortgagors), the empirical models are still widely preferred in practice. Kalotay et al. (2004) discuss this fact together with the shortcomings of previous option-based models in detail and propose a new model which works with two different yield curves for the valuation of a mortgage from a homeowner's perspective and for the valuation of MBS cash flows from an investor's perspective.

In econometric models interest rates are usually simulated under a risk-neutral martingale measure and enter then as explanatory variables into the empirical prepayment model for cash flow projection. The OAS (a constant spread over the benchmark interest rate curve which makes the theoretical price of an empirical valuation model equal to the market price of an MBS) is a common and broadly accepted quantity in the MBS markets, even though its interpretation is subject to discussion. It is a common view among practitioners that the OAS represents a risk premium for prepayment risk. Levin and Davidson (2005) point out, however, that random oscillations of actual prepayments around the model's predictions should be diversifiable and should not lead to any additional risk compensation premium. They thus interpret the OAS as a compensation for non-diversifiable uncertainty which is systematic in trend and unexplained by an otherwise best-guess prepayment model. Kupiec and Kah (1999) argue in a rather similar direction and attribute the existence of the OAS to the omission of important prepayment factors in the risk-neutral Monte-Carlo simulation process. Indeed, in the risk-neutral pricing framework there is no scope for economic risk premia since under a risk-neutral martingale measure all traded assets are expected to earn the risk-free rate.

These recent considerations have directed researchers' attention to the probability measure associated with the prepayment process. Kagraoka (2002) points out that, "surprisingly", this has not been an issue before despite the fact that practitioners have been employing the OAS procedure for decades, emphasizing that it is, of course, not sure that the prepayment process under the martingale measure is equivalent to that under the real world measure. Meanwhile, the recent advances in credit risk modelling have motivated a series of new research papers which are concerned with the valuation of mortgage loans and MBS using approaches borrowed from this field. While Nakagawa and Shouda (2005) use a structural approach in which they define an unobservable prepayment cost process which can be compared to the firm value process in the default risk modelling literature, Goncharov (2005) applies an intensity-based approach as used in reduced-form credit risk models to evaluate mortgage contracts. Kau et al. (2004) use a reduced-form intensity-based approach to model prepayment and default behaviour for individual mortgage contracts in an explicitly specified proportional hazard framework. They also validate their modelling approach empirically by calibrating their model to a large data set of market prices for mortgage contracts.

When making the transition from the valuation of individual mortgage contracts to MBS, the question arises whether all mortgagors in a pool with the same loan characteristics can be assumed to feature independent, homogeneous prepayment and default behaviour. It is a well known fact in the mortgage markets that past refinancing incentives due to low mortgage refinancing rates affect prepayment speeds at pool level in the present and future. This effect is commonly referred to as "burnout" and called "an essential phenomenon of mortgage behaviour" by Levin (2001). Levin's model explicitly separates the pool's mortgagors into an active (ready-to-refinance) and a passive, pure turnover part (including those mortgagors that are not able or not willing to refinance their loans due to, e.g., individual transaction costs or simply lack of financial interest). The traditional way of accounting for burnout is, however, the incorporation of a burnout factor as explanatory variable into the prepayment model as, e.g., in the model developed by Schwartz and Torous (1989). This approach, which we will also take in our model, has the advantage that a rather ad-hoc a priori assumption of mortgagor heterogeneity as in Levin (2001) is not necessary. Apart from Levin and Davidson (2005), where Levin's model is used in a risk-neutral valuation framework for US agency-MBS, work on prepayment-risk-neutral valuation models for MBS is still rare. This is particularly true for work on empirical model validation where the stylized facts commonly observed in the MBS markets, such as the previously described burnout effect or the different OAS levels across different coupons, have to be taken into account.

In this paper we present a prepayment-risk-neutral valuation model for MBS which is based on the proportional hazard model for individual mortgage contracts presented by Kau et al. (2004). However, we use different mean-reverting processes for the interest and baseline prepayment factors and explicitly account for the dependence between baseline turnover prepayment and general economic conditions. This is done by adding a third factor which is fitted to the quarterly GDP growth in the US, making our model a hybrid-form model. Applied to data of GNMA 30yr fixed-rate MBS-pass-throughs our model also allows for a traditional OAS analysis within the same framework. Since detailed mortgage default data is not available, we take an investor's point of view and do not explicitly separate default from prepayment.

The paper is organized as follows: In Section 2 we present our model and provide the necessary mathematical background. Details of the parameter estimation and calibration process are discussed in Section 3. We furthermore present the empirical results and discuss their economic implications. Finally, Section 4 concludes.

# 2 The Model

A crucial part of every valuation model for MBS is an adequate interest rate model. We use a 1-factor Hull-White type model where the non-defaultable short rate $r$ is defined by the dynamics

$$dr(t) = (\theta_r(t) - a_r r(t))dt + \sigma_r dW_r(t). \tag{1}$$

The time-dependent mean-reversion level $\theta_r(t)$ is fitted to the initial term-structure as described in Hull and White (1990). Then, a stochastic prepayment process $p(t)$ is considered in a proportional hazard framework. Corresponding to the model set-up of Kau et al. (2004), the basic idea behind our approach is to capture the turnover component of prepayment in a baseline hazard process, identical for all MBS of the same type, while the pool-specific refinancing components are captured through individual explanatory variables such as the contract rate spread to current mortgage benchmark rates, the pool burnout, etc. Since we find strong empirical evidence for the dependence of the turnover component of prepayment and the quarterly GDP growth in the US (see Section 3) we use a 2-factor model for the baseline hazard process and fit the second factor to the GDP growth data. For both, the baseline hazard, which we incorporate into the overall prepayment process in an exponential way to ensure that prepayment speeds are non-negative, and the general economic conditions represented by the quarterly GDP growth, we assume a mean-reverting process with constant mean-reversion level following Vasicek (1977). Since we only consider 30yr GNMA fixed-rate pass-throughs in this study we assume one common baseline hazard process for all MBS. So, for an MBS with individual covariate vector $\mathbf{x}(t)$ the prepayment processes have the form:

$$\begin{aligned} p(t) &= e^{f(\mathbf{x}(t),\boldsymbol{\beta})+p_0(t)}, && (2)\\ dp_0(t) &= (\theta_p + b_{pw}w(t) - a_p p_0(t))dt + \sigma_p dW_p(t), && (3)\\ dw(t) &= (\theta_w(t) - a_w w(t))dt + \sigma_w dW_w(t), && (4) \end{aligned}$$

where $f(\mathbf{x}(t), \boldsymbol{\beta})$ is some function of the time-dependent covariate vector of the MBS (containing, e.g., contract rate spread and pool burnout) and of the regression parameter vector $\boldsymbol{\beta}$, $p_0$ is the common baseline hazard process, $w(t)$ represents the quarterly US GDP growth and $W_p, W_w$ are independent Wiener processes.

To describe the hazard rate or (instantaneous) prepayment speed $p(t)$ of a mortgage pool we can heavily borrow from intensity-based credit risk modelling approaches as developed, e.g., in Schönbucher (2003). Consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, Q)$ which supports the Wiener processes $W_r, W_p, W_w$ and a counting process $N(t)$, counting the number of mortgages in a pool that have already been prepaid at the point of time $t$. We define $N(t)$ as a doubly stochastic Poisson process, sometimes also called Cox process, with stochastic intensity. Of course, in the Cox Process framework there is no maximum number of jumps, so that we have to assume at this point that there are infinitely many mortgages in a pool. We will come back to this issue later. In addition to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ we consider the filtration $\{\mathcal{G}_t\}_{t\geq 0}$ generated by all of the previously considered processes except the counting process $N(t)$. A filtration of this kind is called "background information filtration" by Schönbucher (2003). We assume that $N(t)$ has a $\{\mathcal{G}_t\}$-measurable intensity $\gamma(t)$ with $\int_0^t \gamma(s)ds < \infty$ for all $t \geq 0$. Then,

$$M(t) := N(t) - \int_0^t \gamma(s)ds$$

is an ($\mathcal{F}_t$-local) martingale and the existence of a unique $\mathcal{F}_t$-predictable version of the intensity $\gamma(t)$ is assured (see Schmid (2004) [Theorems 2.3.1 - 2.3.3] and Brémaud (1981) for details). The expected increment of the Cox process is given by

$$E\left(dN(t)|\mathcal{F}_t\right) = \gamma(t)dt.$$

As a next step, we account for the fact that there are only finitely many mortgages in a pool and approximate $dN(t)$ by $\sum_{k=1}^K dN_k(t)$ where $N_k(t)$ denotes the one-jump prepayment indicator process of the $k$-th mortgage in the pool which, at time t, has a value of 0 if the mortgage has not been prepaid previously, 1 otherwise, and $K$ is the total number of mortgages in the pool. Assuming that the prepayment processes of the individual mortgagors in the pool are homogeneous and independent of each other (an assumption that is maintained at all stages of our modelling approach) it holds that, as $K$ goes to infinity,

$$\frac{1}{K}\sum_{k=1}^{K} dN_k(t) \xrightarrow{\mathcal{D}} \gamma(t)dt, \tag{5}$$

where "$\xrightarrow{\mathcal{D}}$" denotes convergence in distribution. A formal proof of this relation can be found in Kagraoka (2002) as a consequence of the Central

Limit Theorem for Processes (see Jacod and Shiryaev (1987) [VIII 3.46]). Since in this study we are dealing with large mortgage pools of several thousand individual mortgages in each pool we can conclude that

$$p(t)dt = E\left(\frac{1}{K}\sum_{k=1}^{K} dN_k(t)|\mathcal{F}_t\right) \approx \gamma(t)dt. \tag{6}$$

So far, we have only considered the dynamics and properties of the processes under the real-world measure. The key to the transition from the real-world measure $Q$ to an equivalent martingale measure $\widetilde{Q}$ is the Girsanov theorem for marked point processes which is stated in its general version in the Appendix. With a few structural asumptions we can derive the form of the processes (1) to (4) under the equivalent martingale measure, summarized in the following theorem.

**Theorem 1:**
*Let $\phi' = (\phi_r, \phi_p, \phi_w)$ be a three-dimensional predictable process and $\Phi(t)$ a non-negative predictable function with*

$$\int_0^t |\phi_i(s)|^2 ds < \infty, \qquad i = r, p, w, \qquad \int_0^t |\Phi(s)|p(s)ds < \infty$$

*for any finite $t$. Define the process $L$ by $L(0) = 1$ and*

$$\frac{dL(t)}{L(t-)} = \sum_{i=r,p,w} \phi_i(t)dW_i(t) + (\Phi(t) - 1)(dN(t) - p(t)dt).$$

*Assume that $E_Q(L(t)) = 1$ for finite $t$. Define the probability measure $\widetilde{Q}$ with*

$$\left.\frac{d\widetilde{Q}}{dQ}\right| \mathcal{F}_t = L(t), \qquad \forall t \geq 0.$$

*Further assume that there are constants $\lambda_r, \lambda_p, \lambda_w$ such that*

$$\begin{aligned} \phi_r(t) &= \lambda_r \sigma_r r(t) \\ \phi_p(t) &= \lambda_p \sigma_p p_0(t) \\ \phi_w(t) &= \lambda_w \sigma_w w(t) \end{aligned}$$

*and assume that*

$$\Phi(t) = (p(t))^{\mu-1} \tag{7}$$

*for some constant $\mu \in \mathbb{R}$. Then,*

$$\tilde{p}(t) = e^{\mu\cdot(f(\mathbf{x}(t),\boldsymbol{\beta})+p_0(t))}$$

*is the intensity of the counting process $N(t)$ under $\widetilde{Q}$ and the processes (1), (3), (4) have the following dynamics under $\widetilde{Q}$:*

$$\begin{aligned}
dr(t) &= [\theta_r(t) - (a_r + \lambda_r \sigma_r^2) r(t)]dt + \sigma_r d\widetilde{W}_r(t) \\
dp_0(t) &= [\theta_p + b_{pw} w(t) - (a_p + \lambda_p \sigma_p^2) p_0(t)]dt + \\
&\quad + \sigma_p d\widetilde{W}_p(t) \\
dw(t) &= [\theta_w - (a_w + \lambda_w \sigma_w^2) w(t)]dt + \sigma_w d\widetilde{W}_w(t),
\end{aligned}$$

*where $\widetilde{W}_r, \widetilde{W}_p, \widetilde{W}_w$ are independent $\widetilde{Q}$-Wiener processes.*

*Proof.* The Proposition follows from the Girsanov theorem as stated in the Appendix. Recall that our marked point process $N(t)$ is a Cox process with intensity $p(t)$. Thus, the marker space $E$ contains only the element $\{1\}$ and, denoting the marker variable by $Y$, the compensator measure $\nu(de, dt)$ has the form:

$$\nu(de, dt) = \delta_{Y=1}(de) p(t) dt.$$

The Girsanov theorem now yields that (see (16) and (17))

$$\tilde{p}(t) = \Phi(t) p(t)$$

and by the structural assumption (7) we get

$$\tilde{p}(t) = e^{\mu \cdot (f(\mathbf{x}(t), \boldsymbol{\beta}) + p_0(t))}.$$

Furthermore, the Girsanov theorem ensures that, with $d\widetilde{W}_i(t) := dW_i(t) + \phi_i(t)dt$, $\widetilde{W}_i$ is a Wiener-process under $\widetilde{Q}$ for $i = r, p, w$ and we finally get the dynamics of the processes $r(t), p_0(t), w(t)$ under the (martingale) measure $\widetilde{Q}$ by standard argumentation (see, e.g., Zagst (2002) [4.4-4.5] for details). □

Note at this point that the structural assumption (7) is a little unconventional. In most other applications the assumption that $\Phi(t) = \mu^*$ for some constant $\mu^*$ is the norm. In our case, however, it is more convenient to assume a structure as in (7), since this leads to the (multiplicative) risk-adjustment parameter $\mu$ in the overall prepayment process which is clearly identifiable against the risk-adjustment parameters $\lambda_p$ and $\lambda_w$ in the baseline prepayment process $p_0(t)$. The convenience of the previously described structural assumption will become clear in Section 3 where we discuss the interaction of the risk-adjustment parameters in their economic context.

The value $V(0)$ of the MBS at time $t = 0$ can finally be calculated as the expectation of the security's discounted future cash flows under the equivalent martingale measure $\widetilde{Q}$. If we denote by $A(t_k)$ the regular principal amount outstanding on payment date $t_k$ according to the original amortization schedule without any prepayments, by $M$ the original monthly mortgage payment (i.e. the sum of interest and scheduled principal repayment) and by $K$ the number of payment dates until final maturity of the MBS, we get the follwing cash flows at each payment date $t_k$:

- The monthly mortgage payment $M \cdot \prod_{j=1}^{k-1}(1 - p(t_j))$
- The prepaid principal $A(t_k) \cdot p(t_k) \cdot \prod_{j=1}^{k-1}(1 - p(t_j))$.

We can thus conclude:

**Theorem 2:**
*The value $V(0)$ of a fixed-rate MBS at time $t = 0$ is given by:*

$$V(0) = E_{\widetilde{Q}}\left[\sum_{k=1}^{K} c_{t_k}\left(\prod_{j=1}^{k-1}(1 - \tilde{p}(t_j))\right)(\tilde{p}(t_k)A(t_k) + M)\right], \qquad (8)$$

*where* $c_{t_k} = e^{-\int_0^{t_k} r(s)ds}$.

Due to the stochasticity of the prepayment speeds $p(t)$ and the path dependence introduced through the explanatory variables we have no alternative to a computationally costly Monte-Carlo simulation to evaluate (8) at this point.

# 3 Parameter estimation, model calibration and empirical results

## 3.1 Interest rate and real-world prepayment model

The available data for this study consists of US treasury strip par rates and monthly historic prepayment data for large issues of 30yr fixed-rate Mortgage-Backed Securities of the GNMA I and GNMA II programs. We use the historical pool data of a total of eight individual mortgage pools for the empirical prepayment model (see Table 2 for the pool numbers). The corresponding MBS were issued between 1993 and 1996 with more than USD 50m of residential mortgage loans in each of the eight pools and have coupons between 6% and 9%, so that both discounts and premiums are included in our sample. Discount MBS are securities with a low coupon which are traded below 100% while premiums feature high coupons and market prices above 100%. After the months of very high prepayment speeds in 2002-2004 (compare Figure 1) the mortgage pools considered for parameter estimation in this study were not large enough any more to maintain the assumptions based on large sample properties as stated in Section 2. We therefore discard the prepayment data of these pools in 2005 for parameter estimation in the prepayment model. Weekly US treasury strip zero rates, obtained from the par rates by standard bootstrapping, from 1993 to 2005 are used for the estimation of the parameters of the interest rate process. Since the focus of

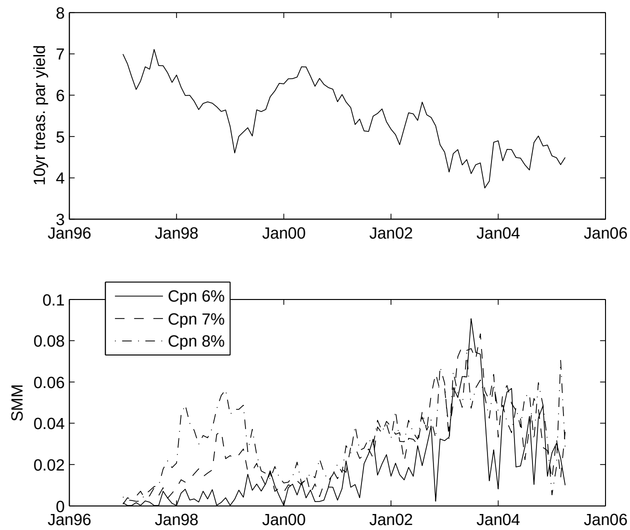


Figure 1: Historical 10yr treasury par yield (top) and prepayment speeds of some selected MBS of the GNMA II program with coupons of 6%, 7% and 8% respectively (bottom).

this paper is not on explanatory variables for prepayment, we restrict the set of covariates to those that are usually stated as the most important ones: the spread between the weigthed-average coupon (WAC) of the mortgage pool and the 10yr treasury par yield which is commonly used as proxy for mortage rates (see, e.g., Goncharov (2005) for some discussion concerning this choice of proxy) and the burnout which we define in line with the definition given in Schwartz and Torous (1989):

$$burnout = ln\left(\frac{AO^*}{AO}\right),$$

where $AO^*$ is the actual principal amount outstanding and $AO$ is the remaining principal amount according to the amortization schedule without any prepayments. In order to account for the usual S-curve shape of the influence of the refinancing incentive, expressed by the spread covariate, we choose the arcus-tangens as functional form. The arcus-tangens function was also used by Asay et al. (1987). Furthermore, our empirical results could be improved by incorporating the burnout covariate as cubic term in addtition to the linear term. Finally, our covariate function $f$ has the form

$$
\begin{aligned}
f(\mathbf{x}(t), \boldsymbol{\beta}) &= \beta_1 \cdot \arctan(\beta_2 \cdot (\mathit{spread}(t) + \beta_3)) + \\
&\quad + \beta_4 \cdot \mathit{burnout}(t) + \beta_5 \cdot \mathit{burnout}(t)^3.
\end{aligned} \tag{9}
$$

We estimate the parameters $a_r, \sigma_r, \lambda_r$ of the interest rate model with a Kalman filter for state space models applying the efficient numerical algorithms described in Koopman et al. (1999) with measurement and transition equations as given in the Appendix.

The results of the parameter estimation for both the interest rate model and the real-world prepayment model are summarized in Table 1. The estimated standard errors of the parameter estimators are obtained by a moving block bootstrapping procedure for which we choose a block length of 100 for the weekly interest rate data and a block length of 20 for the monthly prepayment data. In the block bootstrapping procedure, the blocks are then randomly concatenated to obtain series with the same length as the respective original sample series. The empirical standard deviation of the respective estimator in a total of 50 bootstrap replications yield the standard error estimates as given in Table 1 (see, e.g., Lahiri (2003) for details on block bootstrapping techniques). The estimates of the interest rate model parameters yield an average mean-reversion level of the short-rate of 4.8% (i.e. $\frac{1}{K}\sum_{k=1}^{K} \theta_r(t_k)/a_r = 4.8\%$), which seems to be a fairly appropriate value given that the average observed 3-month rate was 5.1% during the time horizon used for parameter estimation. For the estimation of the prepayment parameters we use a two-stage procedure. We first estimate the parameters $\theta_w, a_w, \sigma_w$ of the GDP growth process by Maximum-Likelihood and the parameters $\theta_p, a_p, \sigma_p, b_{pw}$ again by a Kalman filter for state space models with the historic prepayment speeds as observables (see Appendix). The values for the estimates of the GDP growth process (see again Table 1) yield a mean-reversion level of $\theta_w/a_w = 1.3\%$ which is identical to the actually observed average. In the second stage we then calibrate the prepayment-risk-adjustment parameters $\mu, \lambda_p, \lambda_w$ to market data of GNMA 30yr fixed-rate MBS (see Section 3.2). For all optimization steps we use a combined Downhill-Simplex/Simulated Annealing algorithm as described in Press et al. (1992).

Before we discuss the statistical properties of our estimates in the real-world prepayment state space model and proceed to the calibration and interpretation of the prepayment-risk-adjustment parameters, we want to give some empirical justification for the incorporation of the GDP growth rate as a second factor of the prepayment model. While it is often recognized that the baseline turnover of prepayment is correlated to general economic conditions, nobody (to the authors' best knowledge) has made the effort of explicitly modelling such a dependence structure by considering prepayment

| | Parameter | Estimate | (Std. error) |
|---|---|---|---|
| Short-rate process | $a_r$ | 0.11 | (0.0044) |
| | $\sigma_r$ | 0.0088 | $(8.6 \cdot 10^{-5})$ |
| | $\lambda_r$ | -1380.8 | (48.16) |
| | $h_r$ | 0.0005 | $(1.6 \cdot 10^{-5})$ |
| GDP growth process | $\theta_w$ | 0.019 | (0.0012) |
| | $a_w$ | 1.43 | (0.087) |
| | $\sigma_w$ | 0.002 | $(1.3 \cdot 10^{-5})$ |
| Baseline prepayment process | $\theta_p$ | -3.77 | (0.30) |
| | $a_p$ | 1.20 | (0.064) |
| | $\sigma_p$ | 0.88 | (0.024) |
| | $b_{pw}$ | -88.4 | (22.93) |
| | $h_p$ | 0.70 | (0.005) |
| Regression parameters | $\beta_1$ | 0.67 | (0.10) |
| | $\beta_2$ | 0.92 | (0.24) |
| | $\beta_3$ | -1.55 | (0.12) |
| | $\beta_4$ | 0.003 | (0.013) |
| | $\beta_5$ | 0.007 | (0.0015) |

Table 1: Estimates of the interest-rate model and real-world prepayment model parameters where $h_r$ and $h_p$ are the measurement std. errors of the respective state space models (see Appendix).

jointly with a factor such as the quarterly GDP growth. In order to investigate the value of such an economic factor, we consider the differences $d_i(t)$ between the actual, historically observed prepayment speeds $p_i(t)$ of the $i$-th MBS in the sample and those predicted by the covariates, i.e. not taking into account the baseline hazard in (2):

$$d_i(t) = \ln p_i(t) - f(\mathbf{x}_i(t), \boldsymbol{\beta}), \qquad i = 1, ..., M. \tag{10}$$

We consider the average difference $d(t) := \frac{1}{M}\sum_{i=1}^{M} d_i(t)$ of the $M$ different MBS pools used for prepayment parameter estimation as an estimate for the baseline hazard prepayment process $p_0(t)$ and investigate the correlation between the estimated baseline hazard prepayment and the quarterly GDP growth process (monthly data for the GDP growth process was obtained by cubic spline interpolation). With the MatLab-function *corr*, the Pearson correlation coefficient for a 6-month lag between GDP growth rates and prepayments is estimated as -0.4 with a p-value of 0, clearly rejecting the hypothesis of no correlation. Since a lag of 6 months results in the highest significance level (compared to a lag of 3 and 9 months) we incorporate this time lag into our modelling. Note at this point that, of course, data on GDP growth are published with some delay. We account for this delay, so that, when speaking of a 6-month time lag between GDP growth rates and prepayments, we compare, e.g., prepayments in July with the quarterly GDP growth rate in January of the same year, published a few months later.

The negative sign of the correlation may be surprising at first sight. One possible explanation for this may be the fact that we have not separated prepayment from default. Default is certainly more likely in times of an adverse economic environment with sluggish growth. The time lag of 6 months would suggest that it takes about half a year from a worsening of the general economic conditions to a rise in mortgagors' defaults or simply to a mortgagor's decision to "downsize" a mortgage loan by selling the house and moving to a smaller one which would equally lead to higher prepayment rates. The explicit modelling of default as a separate source of prepayment risk (from a GNMA-investor's point of view) will be the topic of further research.

We finally want to test and verify the statistical assumptions of the prepayment state space model and of the Kalman filtering algorithm. It is essential to assume that the Kalman filter innovations (i.e. the standardized residuals; see, e.g., Schmid (2004) [3.6] for further details) are iid random variables. Furthermore, the model specifications of the state space model require the residuals to be normally distributed with mean 0. To verify these assumptions we apply a couple of tests to the innovations

$$u_t := \frac{\ln p(t) - \ln \hat{p}(t)}{\sqrt{Var(\ln p(t) - \ln \hat{p}(t))}}, \quad t = 1, ..., T$$

| Pool | t-test | Box-Ljung-test | ARCH-test | Lilliefors-test |
|---|---|---|---|---|
| GN 354627 | 0 | 0 | 0 | 1 |
| GN 351408 | 0 | 0 | 0 | 0 |
| GN 352166 | 0 | 0 | 0 | 0 |
| G2 2034 | 0 | 0 | 0 | 1 |
| G2 2054 | 0 | 0 | 0 | 0 |
| G2 2305 | 0 | 1 | 0 | 0 |
| G2 2148 | 0 | 0 | 0 | 1 |
| G2 1856 | 0 | 0 | 0 | 1 |

Table 2: Tests for the hypotheses 1. $E[u_t] = 0$, $\forall t$ (second column), 2. No serial correlation in $(u_t)$ (third column), 3. No first-order heteroscedasticity in $(u_t)$ (fourth column), 4. $(u_t)$ are drawn from a normal distribution (fifth column). A value of 0 indicates that the respective hypothesis can not be rejected at the 5% level, a value of 1 indicates that the hypothesis can be rejected at the 5% level.

of our model where $\hat{p}(t)$ is the prepayment speed predicted by our Kalman filter. First of all, we test the hypothesis that $E[u_t] = 0$, $t = 1, ..., T$, with a simple t-test. We then test for serial correlation by applying the Box-Ljung test (see Ljung and Box (1978)). First-order heteroscedasticity is tested with the ARCH-test, which goes back to Engle (1982). We finally apply the Lilliefors-test to test the Normal distribution asssumption (see Lilliefors (1967)). We use the Matlab implementation of these tests and apply them to each of the eight mortgage pools whose prepayment history we use for parameter estimation. Table 2 shows the results from which we can conclude that, altogether, the assumptions of the Kalman filter algorithm are sufficiently satisfied for our prepayment data.

### 3.2 Prepayment-risk-neutral model

In general, there is no active and liquid market for an individual mortgage pool. One obvious reason for this is the limited size of individual pools. For the calibration of the prepayment-risk-adjustment parameters we therefore consider market prices of generic GNMA 30yr fixed-rate MBS as quoted in Bloomberg (Bloomberg ticker GNSF) for trading on a TBA (to-be-announced) basis. We consider coupons between 4.5% and 8%, so that both discounts and premiums are included. GNMA securities which are traded on a TBA basis are highly liquid securities, so that we do not have to worry about liquidity effects/premia. Even for those securities with a coupon well below or well above the current coupon, Bid-Ask spreads are usually not higher than 2 ticks with a tick size of USD $\frac{1}{32}$. When we speak of market prices we refer to the Ask-prices.

We estimate the prepayment-risk-adjustment parameters $\mu, \lambda_p, \lambda_w$ by minimizing the norm of the vector of differences between the market prices and model prices of the securities on each sample day. All calculations are carried out without accounting for any OAS, i.e. with an OAS equal to 0 for all securities. By setting the OAS-target equal to 0, we price with the treasury curve as benchmark curve, which seems to be the most appropriate curve for GNMA securities since these securities feature the full faith and credit of the US Government. Of course, any other curve could be used as benchmark curve if desired. Once the parameters have been calibrated, one can hardly expect all theoretical prices to match market prices exactly for all MBS securities. The OAS equivalent in a risk-neutral valuation framework (with a target of 0 in the calibration procedure) could be compared to the "prOAS"-measure recently introduced by Levin and Davidson (2005). We will also use their "prOAS"-term in the following and point out that the prOAS should not be regarded as any kind of risk premium, but simply as a measure of unsystematic residual pricing error.

Levin and Davidson (2005) emphasize the necessity of a two-risk factor model in order to account for the two distinct market fears in the MBS market: refinancing understatement and turnover overstatement. These two distinct market fears explain why, in the traditional OAS valuation approach, it is not uncommon to observe higher OAS levels for both discounts and premiums compared to MBS around the current-coupon level. On the one hand, an investor in discounts experiences losses if the turnover component is overestimated and pure turnover-related prepayment is slower than expected. In this case the average life of the security is extended, decreasing the cash flow stream's present value. On the other hand, the refinancing component is the major concern of an investor in premiums since the average life of premiums decreases if refinancing-related prepayment is faster than originally estimated, pulling the security's present value towards 100%. This would evidently result in a loss for the holder of a premium MBS.

These considerations are fully accounted for in our model since we have the (multiplicative) risk-adjustment parameter $\mu$ and the two (additive) risk-adjustment parameters $\lambda_p$ and $\lambda_w$. For parameter values of $\mu$ larger than 1, the refinancing S-curve is stretched, i.e. the prepayment incentive induced by higher spreads between the WAC and the 10yr treasury par yield is accelerated. The parameters $\lambda_p$ and $\lambda_w$ only affect the baseline prepayment speed. Note that the mean-reversion level of the Vasicek process for $p_0(t)$ is negative in real-world when we set the GDP growth process $w(t)$ to its mean-reversion level. Thus, for values of $\mu$ larger than 1 (as in our estimates in Table 3), the process $p_0(t)$ will take much smaller values (larger in absolute terms), potentially reducing the overall prepayment speed for both discounts and premiums. Now, for positive values of $\lambda_p$ and (with much less significant consequences) $\lambda_w$, the mean-reversion level of the process $p_0$ will be pulled back into the positive direction, bringing back the overall prepayment speed to sensible levels for discounts and premiums in the same (additive) way. With our structural assumption for the prepayment intensity under the risk-neutral measure as given in (7) we can therefore accelerate prepayments for premiums while, at the same time, decelerate prepayments for discounts under the risk-neutral measure. We can thus account for both, the market fear of turnover overstatement for discounts and the market fear of refinancing understatement for premiums, in our prepayment-risk-neutral pricing approach.

Figure 2 illustrates how the prepayment-risk-adjustment parameters account for the two types of prepayment risk as previously discussed. Under the risk-neutral measure prepayment speeds are slower for low spreads which extends the average life of discounts. Contrarily, prepayment speeds for high spreads rise under the risk-neutral measure, shortening the average life of premiums and thus clearly reflecting the market fear of refinancing understatement.

Note also at this point that a traditional OAS valuation is of course easy to perform within our modelling framework by simply setting $\mu = 1$

| Parameter | 18-Oct-2005 | 04-Nov-2005 | 12-Dec-2005 |
|---|---|---|---|
| $\mu$ | 2.2 | 2.4 | 2.0 |
| $\lambda_p$ | 2.7 | 3.0 | 2.2 |
| $\lambda_w$ | -10.2 | -8.1 | -7.0 |

Table 3: Estimates of the prepayment-risk-adjustment parameters on three (arbitrarily chosen) dates.

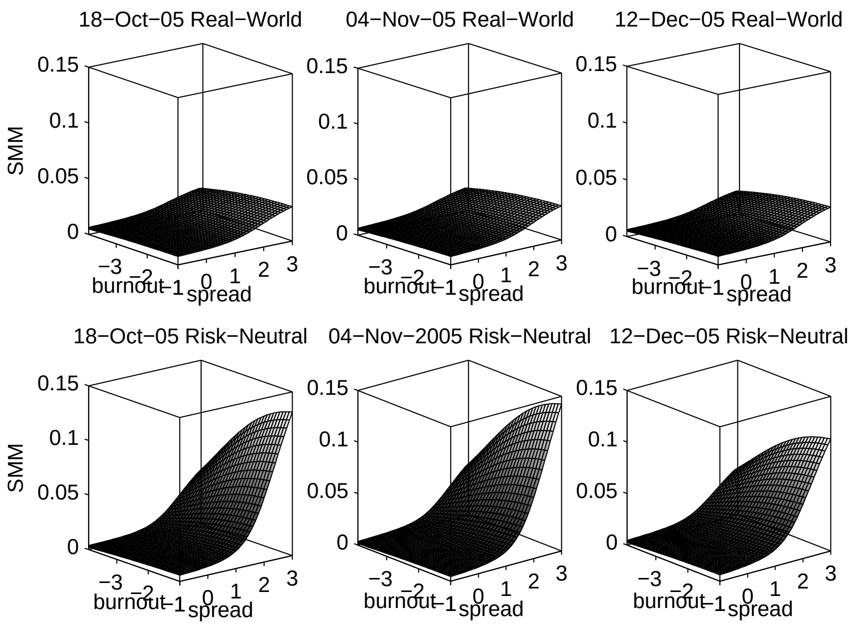


Figure 2: Expected prepayment speed as a function of the two covariates *spread* (in %) and *burnout* in real-world and under the risk-neutral measure when the baseline hazard process is set equal to its mean-reversion level.

and $\lambda_p = \lambda_w = 0$. In this case prepayments would be forecast under the real-world measure and the OAS would be needed to equate the model prices to the observed market prices. Figure 3 shows the traditional OAS and the prOAS values of our model and for our sample of GNMA securities. On each of the three days the current coupon was between 5.5% and 6%. For comparison purposes we also show the OAS levels as quoted in Bloomberg based on the Bloomberg prepayment model. Of course, it is hard to compare OAS levels derived from different prepayment models. As already discussed in Kupiec and Kah (1999), it is very common in the MBS markets that OAS estimates of different brokers vary widely, attributable to different interest

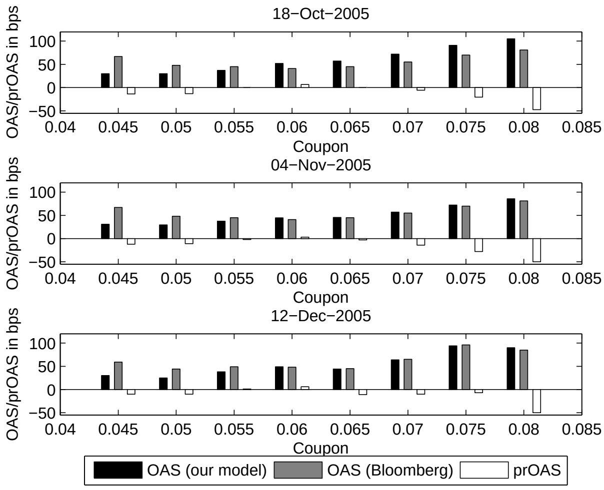


Figure 3: OAS according to our model, OAS as quoted in Bloomberg and "prOAS" according to our model on 18-Oct-2005 (top), 04-Nov-2005 (centre) and 12-Dec-2005 (bottom) for a series of generic 30yr fixed-rate MBS of the GNMA I program with different coupons (Bloomberg ticker GNSF).

rate and prepayment model assumptions. This fact may provide a further line of argumentation for prepayment-risk-neutral models like the one presented in this paper. In addition to the prOAS levels in Figure 3 we also show the market prices of the GNMA securities in our sample directly compared to the risk-neutral model prices in Figure 4. These plots confirm that, generally, our model successfully explains market prices of generic fixed-rate GNMA pass-throughs. Note that this is also true if we calibrate the risk-adjustment parameters only once to the data on 18-Oct-2005 and leave the parameters unchanged for our additional sample dates 04-Nov-2005 and 12-Dec-2005. As a quantitative measure of the accuracy of our pricing approach we consider the linear regression model

$$V_i^{market} = a + b \cdot V_i^{model} + \epsilon_i, \quad \epsilon_i \overset{iid}{\sim} N(0, \sigma_\epsilon^2), \quad i = 1, ...I \qquad (11)$$

where $V^{market}$ denotes the market prices of the MBS, $V^{model}$ the prices of the MBS according to our prepayment-risk-neutral valuation model and $I = 24$ is the total number of observed market prices in our sample (we consider 8 securities on 3 different days). Obviously, the estimates of the regression parameters $a$ and $b$ should be close to 0 and 1 respectively. The actual estimates together with the $R^2$ value of the regression are reported in Table 4 in comparison to the values which we obtain with a 1-factor baseline model (i.e. without the GDP factor). Since in the 2-factor baseline prepayment model the confidence intervals for $a$ and $b$ are narrower around

0 and 1 respectively and the $R^2$ value is higher, these results indicate that the GDP growth factor adds explanatory power to our prepayment-risk-neutral pricing model. The (in-sample) average absolute pricing error of our model, i.e. the mean of the absolute differences between the model prices and the market prices, is 59 basis points in our sample compared to 105 basis points for the 1-factor baseline prepayment model. When we consider out-of-sample prices, i.e. we use the risk-adjustment parameters calibrated to the data of 18-Oct-2005 for pricing on the two other sample days, we obtain an average absolute pricing error of 61 basis points for our model while the average absolute pricing error of the 1-factor baseline prepayment model is 76 basis points. These results provide further evidence for the usefulness of the GDP growth rate factor.

| | Parameter | Estimate | 95%-Conf.Int. |
|---|---|---|---|
| Regression (11) with 2-factor baseline prepayment model $R^2 = 98.5\%$ | $a$ | 0.023 | [-0.030;0.077] |
| | $b$ | 0.982% | [0.929;1.035] |
| | | | |
| Regression (11) with 1-factor baseline prepayment model $R^2 = 96.2\%$ | $a$ | 0.0069 | [-0.082;0.096] |
| | $b$ | 0.985 | [0.898;1.072] |
| | | | |

Table 4: Parameter estimates and $R^2$ of the regression model (11) when the model prices are calculated with the 2-factor baseline prepayment model and when the model prices are calculated with a 1-factor baseline prepayment model without the GDP growth process.

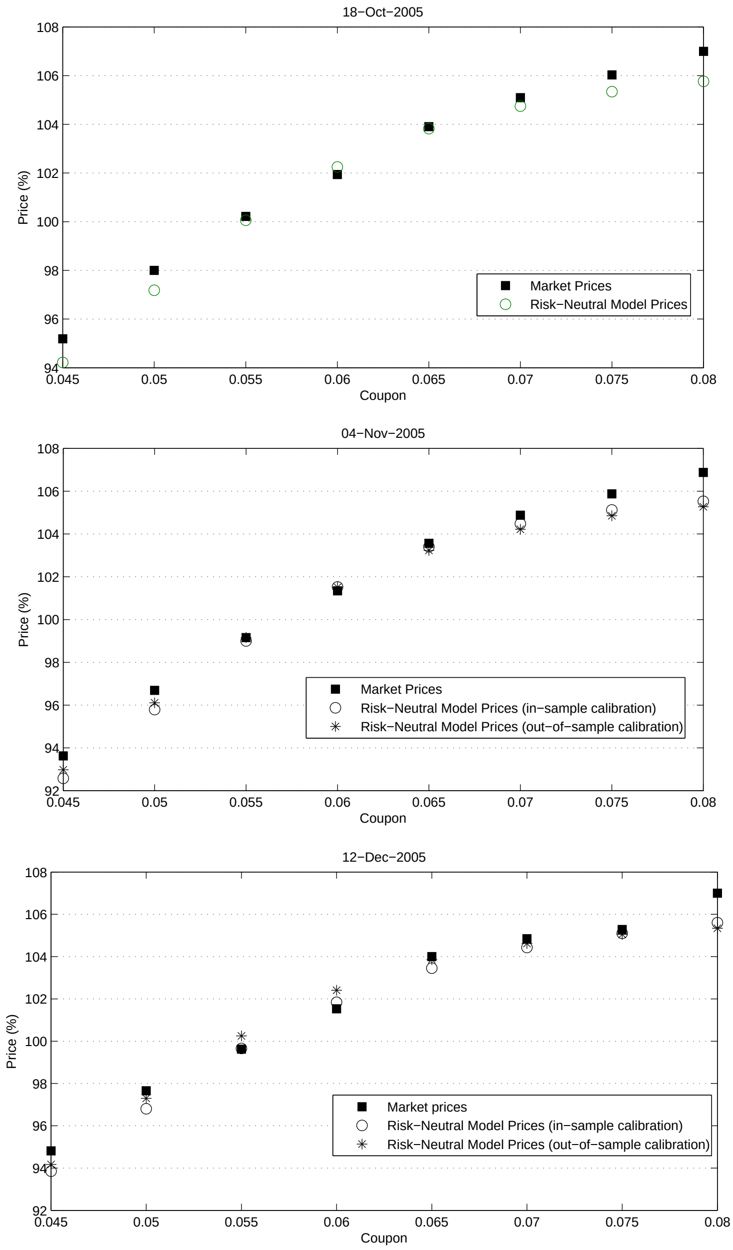


Figure 4: Market prices and model prices on 18-Oct-2005 (top), 04-Nov-2005 (centre) and 12-Dec-2005 (bottom) for a series of generic 30yr fixed-rate MBS of the GNMA I program with different coupons (Bloomberg ticker GNSF). Out-of-sample calibration means that we do not recalibrate the risk-adjustment parameters on the respective day, but use the parameter values of 18-Oct-2005 instead.

# 4 Conclusion

In this paper we have presented a prepayment-risk neutral valuation model for fixed-rate Mortgage-Backed Securities. Based on reduced-form intensity models in a proportional hazard framework, our model explicitly accounts for the general economic environment by the incorporation of a factor which is fitted to the GDP growth rate. Applied to a series of GNMA MBS with different coupons we were able to successfully explain market prices across the different coupons. While recognizing that a 1-factor model for the baseline prepayment process in the spirit of Kau et al. (2004) is not statistically misspecified, we have found that the GDP factor adds explanatory power to our model when applied to the market prices which were available for this study. Both the risk of refinancing understatement (for premiums) and the risk of turnover overstatement (for discounts) are accounted for in our prepayment-risk-neutral setting by three prepayment-risk adjustment parameters.

Since the focus of the paper is not on covariates we have restricted our attention to the contract rate spread and to a factor accounting for the burnout effect. However, our modelling framework is of course flexible enough to incorporate any other exogenously given covariate, e.g. ramp period effects, average loan size or geographical information, in a straightforward way.

In models like the one presented in this paper it is also important to get a feeling of how often parameters should be re-calibrated. This is especially true for the risk-adjustment parameters since they represent changing perceptions of risk over time. In our data sample we have seen that the risk-adjustment parameters do not change substantially within a couple of weeks. In addition to this, even by leaving the risk-adjustment parameters unchanged, we have seen in our sample that our model still predicts market prices with a very acceptable accuracy. Yet, a systematic long-term study over a whole economic cycle and with more data would of course be interesting. For now we can conclude that a re-calibration of the risk-adjustment parameters every couple of weeks should be sufficient for most purposes.

# A Parameter estimation

## A.1 Parameter estimation for the short-rate process

The price of a zero-coupon bond with maturity $T$ at the point of time $t$, denoted by $P(t,T)$, in the Hull-White type short-rate model (1) is given by (see, e.g., Zagst (2002)):

$$\begin{aligned} P(t,T) &= e^{A(t,T)-B(t,T)r(t)}, \\ A(t,T) &= \int_t^T \left(\frac{1}{2}\sigma_r^2 B(l,T)^2 - \theta_r(l)B(l,T)\right)dl, \\ B(t,T) &= \frac{1}{\hat{a}_r}(1-e^{-\hat{a}_r(T-t)}), \end{aligned}$$

where $\hat{a}_r = a_r + \lambda_r \cdot \sigma_r^2$. At the point of time $t_k$ we observe for maturities $\tau_i$, $i=1,...,n$, the treasury strip rates $R(t_k, t_k+\tau_i) = -\frac{lnP(t_k,t_k+\tau_i)}{(T-t)}$. With $a(t,T) = -\frac{A(t,T)}{T-t}$, $b(t,T) = \frac{B(t,T)}{T-t}$ the measurement equation of the state space model is then given by:

$$\begin{pmatrix} R(t_k, t_k+\tau_1) \\ \vdots \\ R(t_k, t_k+\tau_n) \end{pmatrix} = \begin{pmatrix} a(t_k, t_k+\tau_1) \\ \vdots \\ a(t_k, t_k+\tau_n) \end{pmatrix} + \begin{pmatrix} b(0,\tau_1) \\ \vdots \\ b(0,\tau_n) \end{pmatrix} \cdot r(t_k) + \epsilon_k, \quad (12)$$

where we assume that the measurement error follows an n-dimensional Normal distribution with expectation vector $\mathbf{0}$ and covariance matrix $h_r^2 \cdot \mathbf{I}_n$, i.e. $\epsilon_k \sim N_n(\mathbf{0}, h_r^2 \cdot \mathbf{I}_n)$. The transition equation can be derived by the dynamics given by the SDE (1). Since a linear SDE

$$dX(t) = (H \cdot X(t) + J(t))dt + VdW(t) \quad (13)$$

with an $m$-dimensional stochastic process $X$, $H \in \mathbb{R}^{m\times m}, V \in \mathbb{R}^{m\times m}$, $J : [0,\infty)^m \to \mathbb{R}^m$ continuous, has the unique strong solution

$$X(t) = e^{H\cdot t}X(0) + \int_0^t e^{H\cdot(t-l)}J(l)dl + \int_0^t e^{H\cdot(t-l)}VdW(l)$$

(see, e.g., Karatzas and Shreve (1997) [5.6]), we get, by defining $X(t) = r(t)$, $H = -a_r$, $J(t) = \theta_r(t)$, $V = \sigma_r$, $r_k = r(t_k)$ and $\Delta t_{k+1} = t_{k+1} - t_k$:

$$r_{k+1} = e^{-a_r\cdot\Delta t_{k+1}}r_k + \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)}\theta_r(l)dl + \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)}\sigma_r dW(l).$$

By approximating $\theta_r(l)$ by $\theta_r(t_k)$ in the integral and defining

$$\eta_{k+1} := \int_{t_k}^{t_{k+1}} e^{-a_r(t_{k+1}-l)}\sigma_r dW(l)$$

we finally get the transition equation of the state space model:

$$r_{k+1} = e^{-a_r \cdot \Delta t_{k+1}} r_k + \int_0^{\Delta t_{k+1}} e^{-a_r \cdot l} \theta_r(t_k) dl + \eta_{k+1} \tag{14}$$

with

$$\eta_{k+1} \sim N_1 \left( 0, \frac{\sigma_r^2}{2a_r}(1 - e^{-2a_r \Delta t_{k+1}}) \right).$$

## A.2 Parameter estimation for the GDP growth process

The dynamics of the GDP growth process in (4) are again given by a SDE of the form (13). Thus, we get

$$\begin{aligned} w(t_{k+1}) &= e^{-a_w \Delta t_{k+1}} w(t_k) + \int_{t_k}^{t_{k+1}} e^{-a_w(t_{k+1}-l)} \theta_w dl + \\ &\quad + \int_{t_k}^{t_{k+1}} e^{-a_w(t_{k+1}-l)} \sigma_w dW_w(l) \end{aligned}$$

and it follows that

$$\begin{aligned} w(t_{k+1})|w(t_k) &\sim N_1(c, d^2), \\ c &= e^{-a_w \Delta t_{k+1}} w(t_k) + \frac{\theta_w}{a_w}(1 - e^{-a_w \Delta t_{k+1}}), \\ d^2 &= \frac{\sigma_w^2}{2a_w}(1 - e^{-2a_w \Delta t_{k+1}}). \end{aligned}$$

We obtain Maximum-Likelihood estimates of the parameters $\theta_w, a_w, \sigma_w$ by maximizing the likelihood function

$$L(\theta_w, a_w, \sigma_w) = \prod_{k=1}^{K} \varphi_{w(t_k)|w(t_{k+1})},$$

where $\varphi_{w(t_k)|w(t_{k+1})}$ denotes the p.d.f. of the Normal distribution with parameters $c$ and $d^2$ as defined above.

## A.3 Parameter estimation for the prepayment process

The measurement equation of the state space model is given by (2) with the historically observed prepayment speeds (as SMM) and $f$ as specified in (10):

$$\begin{pmatrix} \ln(p_1(t_k)) \\ \vdots \\ \ln(p_N(t_k)) \end{pmatrix} = \begin{pmatrix} f(\mathbf{x}_1(t_k), \boldsymbol{\beta}) \\ \vdots \\ f(\mathbf{x}_N(t_k), \boldsymbol{\beta}) \end{pmatrix} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \cdot p_0(t_k) + \boldsymbol{\epsilon}_k, \tag{15}$$

where we assume that $\boldsymbol{\epsilon}_k \sim N_N\left(\mathbf{0}, h_p^2 \cdot \mathbf{I}_N\right)$. The transition equation for the (unobservable) baseline prepayment hazard is given by (3). For stability reasons, we use $w(t)$ as an external input to the model and define $X(t) = p_0(t)$, $H = -a_p$, $J(t) = \theta_p + b_{pw}w(t)$, $V = \sigma_p$ to get a SDE of the form (13). Similar to the derivation of the transition equation (14) of the interest rate model the transition equation of the prepayment state space model is

$$p_0(t_{k+1}) = e^{-a_p \Delta t_{k+1}} \cdot p_0(t_k) + \frac{\theta_p + b_{pw}w(t_k)}{a_p} \cdot (1 - e^{-a_p \Delta t_{k+1}}) + \eta_{k+1}$$

with

$$\eta_{k+1} \sim N_1\left(0, \frac{\sigma_p^2}{2a_p}(1 - e^{-2a_p \Delta t_{k+1}})\right).$$

# B The Girsanov theorem for marked point processes

**Theorem B.1:**
*Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, Q)$ be a filtered probability space which supports an $n$-dimensional $Q$-Wiener process $W(t)$ and a marked point process with jump measure $\eta(de, dt)$. The marker $e$ of the marked point process is drawn from the mark space $(E, \mathcal{E})$. The compensator of $\eta(de, dt)$ is assumed to take the form $\nu^Q(de, dt) = K^Q(t, de)\gamma^Q(t)dt$ under $Q$. Here $\gamma^Q(t)$ is the $Q$-intensity of the arrivals of the point process and $K^Q(t, de)$ is the $Q$-conditional distribution of the marker on $(E, \mathcal{E})$.*
*Let $\phi$ be an $n$-dimensional predictable process and $\Phi(e, t)$ a non-negative predictable function with*

$$\int_0^t |\phi_i(s)|^2 ds < \infty, \qquad \int_0^t \int_E |\Phi(e, s)| K^Q(t, de)\gamma^Q(s) ds < \infty$$

*for any finite $t$. Define the process $L$ by $L(0) = 1$ and*

$$\frac{dL(t)}{L(t-)} = \phi(t) dW(t) + \int_E (\Phi(e, t) - 1)(\eta(de, dt) - \nu^Q(de, dt)).$$

*Assume that $E_Q(L(t)) = 1$ for finite $t$. Define the probability measure $\widetilde{Q}$ with*

$$\left.\frac{d\widetilde{Q}}{dQ}\right| \mathcal{F}_t = L(t), \qquad \forall t \geq 0.$$

*Then:*

1. *The process $\widetilde{W}$ is a $\widetilde{Q}$-Wiener process where $\widetilde{W}(0) = 0$ and*

$$d\widetilde{W}(t) := dW + \phi(t) dt$$

2. *The predictable compensator of $\eta$ under $\widetilde{Q}$ is*

$$\nu^{\widetilde{Q}}(de, dt) = \Phi(e,t)\nu^{Q}(de, dt) \tag{16}$$

3. *Define $\mu(t) := \int_E \Phi(e,t)K^Q(t, de)$ and $L_E(e,t) = \frac{\Phi(e,t)}{\mu(t)}$ for $\mu(t) > 0$, $L_E(e,t) = 1$ otherwise. The intensity of the counting process of the arrivals of the marked point process under $\widetilde{Q}$ is*

$$\gamma^{\widetilde{Q}}(t) = \mu(t)\gamma^{Q}(t) \tag{17}$$

4. *The conditional distribution of the marker under $\widetilde{Q}$ is*

$$K^{\widetilde{Q}}(t, de) = L_E(e,t)K^Q(t, de).$$

*Proof.* See Schönbucher (2003) [4.11] and Jacod and Shiryaev (1987) [III.3]. $\square$

# References


Asay, M., F. Guillaume, and R. Mattu (1987). *Duration and Convexity of Mortgage-Backed Securities: Some Hedging Implications from a Simple Prepayment-Linked Present-Value Model,* Mortgage-Backed Securities: New Strategies, Applications and Research (Fabozzi, F., Ed.)*, 103-125.* Chicago: Probus.

Brémaud, P. (1981). *Point Processes and Queues.* New York: Springer.

Engle, R. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of UK Inflation. *Econometrica 50*, 987–1008.

Goncharov, Y. (2005). An Intensity-Based Approach to the Valuation of Mortgage Contracts and Computation of the Endogenous Mortgage Rate. *To appear in: International Journal of Theoretical and Applied Finance*.

Hull, J. and A. White (1990). Pricing interest rate derivative securities. *Review of Financial Studies 3-4*, 573–592.

Jacod, J. and A. Shiryaev (1987). *Limit Theorems for Stochastic Processes.* Berlin: Springer Verlag.

Kagraoka, Y. (2002). The OAS Approach and the Martingale Measure for Mortgage Prepayment. *2nd World Congress of the Bachelier Finance Society*.

Kalotay, A., D. Yang, and F. Fabozzi (2004). An Option-Theoretic Prepayment Model for Mortgages and Mortgage-Backed Securities. *International Journal of Theoretical and Applied Finance 7*, 949–978.

Karatzas, I. and S. Shreve (1997). *Brownian Motion and Stochastic Calculus (2nd edition)*. New York: Springer.

Kau, J., D. Keenan, and A. Smurov (2004). Reduced-Form Mortgage Valuation. *Working Paper*.

Koopman, S., N. Shephard, and J. Doornik (1999). Statistical algorithms for models in state space using Ssfpack 2.2. *Econometrics Journal 2*, 113–166.

Kupiec, P. and A. Kah (1999). On the Origin and Interpretation of OAS. *The Journal of Fixed Income 9-3*, 82–92.

Lahiri, S. (2003). *Resampling Methods for Dependent Data*. New York: Springer.

Levin, A. (2001). Active-Passive Decomposition in Burnout Modeling. *The Journal of Fixed Income 10-4*, 27–40.

Levin, A. and A. Davidson (2005). Prepayment Risk- and Option-Adjusted Valuation of MBS - Opportunities for Arbitrage. *The Journal of Portfolio Management 31-4*.

Lilliefors, H. (1967). On the Kolmogorov-Smirnov test for normality with mean and variance unknown. *Journal of the American Statistical Association 62*, 399–402.

Ljung, G. and G. Box (1978). On a Measure of Lack of Fit in Time Series Models. *Biometrika 65-2*, 297–303.

Nakagawa, H. and T. Shouda (2005). Analyses of Mortgage-Backed Securities Based on Unobservable Prepayment Cost Processes. *Working Paper*.

Press, W., S. Teukolsky, W. Vetterling, and B. Flamery (1992). *Numerical Recipes in C, The Art of Scientific Computing (2nd edition)*. Cambridge: Cambridge University Press.

Schmid, B. (2004). *Credit Risk Pricing Models: Theory and Practice (2nd edition)*. Berlin: Springer Finance.

Schönbucher, P. (2003). *Credit derivatives pricing models*. Chichester: John Wiley & Sons.

Schwartz, E. and W. Torous (1989). Prepayment and the Valuation of Mortgage-Backed Securities. *The Journal of Finance 44*, 375–392.

Stanton, R. (1995). Rational Prepayment and the Valuation of Mortgage-Backed Securities. *Review of Financial Studies 8*, 677–708.

Vasicek, O. (1977). An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics 5*, 177–188.

Zagst, R. (2002). *Interest Rate Management*. Berlin: Springer Finance.