Valuation of Mortgage-Backed Securities and Mortgage Derivatives: A Closed-Form Approximation

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Abstract

In this paper we develop a closed-form and thus computationally highly efficient formula to approximate the value of fixed-rate MBS. Our modelling framework is based on reduced-form and prepayment-risk neutral valuation techniques and offers two major extensions compared to existing closed-form approximation approaches: We are not limited to one stochastic factor and we are able to capture the usual S-curve shape of the refinancing incentive by a piecewise linear approximation. We apply our model to a sample of GNMA pass-through securities and find that our model is able to explain market price movements with a highly satisfactory accuracy for a wide range of coupons.

Keywords: mortgage-backed security, prepayment, closed-form, risk-neutral pricing

JEL classification: G12, G13

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1 Introduction

The valuation of mortgage-backed securities is usually considered a computationally expensive problem. This holds for both categories of valuation models that have been discussed in the academic and practitioner-oriented literature so far. The first model category is usually called the option-based approach where prepayment is related to a mortgagor's rational decision to exercise the prepayment option inherent in the mortgage contract. In this class of models the modelling of prepayment and the valuation of MBS is closely related to the pricing of American-style interest-rate options and of callable bonds. Hence, most authors use backward induction valuation approaches on multidimensional grids to solve the partial differential equation (PDE) which the mortgage/MBS value must satisfy (see, e.g., Stanton (1995), Kalotay et al. (2004) or Sharp et al. (2006)). The grid dimension is determined by the number of factors which enter into the prepayment modelling. Thus, the grid points grow exponentially with the number of factors, making most numerical PDE methods ponderously slow. The second class of prepayment models for MBS is usually labelled econometric models where an empirically estimated prepayment function, often within a proportional hazard framework, is used to forecast prepayment cash flows. The computational burden for the pricing of MBS is particularly high for the econometric models where a computationally expensive Monte-Carlo simulation is the only possibility in most cases (see, e.g., Schwartz and Torous (1989) or the modelling approach in Kolbe and Zagst (2006)).

The computational burden can constitute a serious problem, particularly when dealing with large portfolios of MBS which have to be revaluated frequently, e.g. in a risk or portfolio management context. Yet, in such an environment, a fast-to-compute closed-form approximation of a security's value would be sufficient for most purposes. This fact was also noted by Collin-Dufresne and Harding (1999) and Sharp et al. (2006), the only two papers so far (to the authors' best knowledge) which are concerned with closed-form formulas for mortgages or MBS. The latter, however, only addresses the valuation of a single fixed-rate mortgage contract by a purely option-theoretic approach for which a closed-form approximation is derived by the use of singular perturbation theory for PDEs. A generalization of this approach to the valuation of MBS may not be straightforward due to the non-optimal and heterogeneous prepayment behaviour of the different mortgagors in a MBS pool. These are the typical shortcomings of option-based approaches which are discussed in detail in Kalotay et al. (2004). Most option-theoretic models have not been able to explain observed market prices of MBS consistently so far. In particular, they are often not able to take into account the stylized facts which can commonly be observed in the MBS markets. This often results in inconsistent option-adjusted spread (OAS) patterns across the different coupons. The OAS is a constant spread which is added to each discount rate so that model prices equal market prices. For this reason econometric models with an empirically estimated prepayment function are still widely preferred in practice.

In the paper by Collin-Dufresne and Harding (1999) a combination of the two different approaches is used: An empirically estimated prepayment function is incorporated into an option-based approach. While with this model the authors are able to explain most of the historic price variation of an exemplarily chosen security, their model has a couple of shortcomings. First, their modelling framework is limited to one stochastic factor (the risk-free short rate). Second, the relation between interest rates and prepayments is strictly linear, which is not in line with the empirically well established S-curve shape of the refinancing incentive (see, e.g., Levin and Daras (1998)). Finally, their model does not allow for any path-dependent explanatory variables such as the burnout effect. The burnout effect reflects the fact that mortgage pools with few remaining mortgages in the pool usually feature slower prepayments than comparable 'fresh' pools.

In this paper we develop a closed-form formula for the value of fixed-rate (agency) MBS and, as corollaries, for Interest-Only and Principal-Only securities. The valuation is based on a third class of modelling approach which has been adapted to MBS modelling from the credit risk literature recently (e.g. by Goncharov (2005)) and is commonly referred to as reduced-form approach. Our basic model set-up goes back in spirit to the reduced-form model for individual mortgage contracts presented by Kau et al. (2004). In many reduced-form modelling approaches for the valuation of defaultable bonds and/or credit derivatives closed-form pricing formulas are available. This idea, however, has not been applied (to the authors' best knowledge) to prepayment-sensitive MBS. Interestingly, in our reduced-form model we find that a closed-form solution of the MBS valuation problem leads to rather similar challenging calculations as in the option-based approach presented by Collin-Dufresne and Harding (1999). In our framework, however, we are able to address two of the three previously mentioned shortcomings which may be problematic in certain situations and for certain types of MBS. While path-dependencies in a pool's prepayment behaviour can only be modelled up to some reasonable deterministic approximation, it is straightforward to incorporate additional stochastic factors into our model. We do this by modelling the (non refinancing-related) baseline prepayment process via two stochastic factors, where the second factor is fit to the GDP growth in the US. We thus account for the dependence between general economic conditions and turnover-related prepayment in our model. The baseline prepayment is also supposed to capture defaults which, in the case of agency MBS, simply result in prepayment for an investor. In addition to this, we account for the usual S-curve shape of the refinancing-incentive/prepayment relation by a sectionwise linear approximation. We find that this approximation does have an important effect across the whole coupon range.

While in our modelling framework it is straightforward to conduct a classical OAS valuation, we are primarily interested in a prepayment-risk-neutral valuation. This approach directly targets market prices in the spirit of Levin and Davidson (2005) and Kolbe and Zagst (2006). The prepayment-risk-neutral valuation principle also allows us to assess the performance of our model. This is difficult in the classical OAS framework since it is very common that OAS levels of different brokers in the MBS market vary widely due to different interest-rate and prepayment-model assumptions (see Kupiec and Kah (1999)).

The paper is organized as follows: We present our model in Section 2 and derive the closed-form formula step-by-step with all necessary theoretical details. Parameter estimation and model calibration are described in Section 3. The performance and adequacy of our modelling approach is assesed in Section 4 where we apply our model to market data of a series of GNMA fixed-rate MBS. Finally, Section 5 concludes.

2 The Model

As previously mentioned, our pricing formula builds on approaches and pricing formulas from the reduced-form credit risk literature. Our starting point is the valuation of a single mortgage contract. We assume that the time of prepayment of one mortgage does not influence the probability of prepayment of other mortgages and that the pool is homogeneous (w.r.t. mortgage maturity, coupon, etc. and thus w.r.t. individual prepayment probabilities). Thus, the value of the MBS is just the sum of the values of the individual mortgages in the pool. While this assumption is problematic in option-based models which rely on optimal exercise strategies of the prepayment option, this is not the case in the reduced-form framework. We further assume that partial prepayment is not possible.

Consider a mortgage contract with payment dates $t_1, ..., t_K$, define $\Delta t_k := t_k - t_{k-1}$ (years) and set $t_0 = 0$. On each payment date t_k, $k = 1, ..., K$, the mortgage payment $M \cdot \Delta t_k$, containing both interest and regular repayments, has to be made until the time of prepayment. We assume that prepayment is only possible on the regular payment dates t_k. At the time of prepayment t_τ (or at the final maturity of the mortgage), the remaining principal balance according to the amortization schedule $A(t_\tau)$ is paid back in a lump sum. Thus, apart from the fixed payment $M \cdot \Delta t_1$ on the first payment date, all further cash flows depend on the time of prepayment. The value $V(0)$ of the mortgage contract at time 0 admits the representation

$$\begin{aligned} V(0) \quad = \quad & E_{\widetilde{Q}}\left[M \cdot \Delta t_1 \cdot e^{-\int_0^{t_1} r(s)ds} + A(t_1) \cdot p(t_1) \cdot \Delta t_1 \cdot e^{-\int_0^{t_1}(r(s)+p(s))ds}\right] \\ & + E_{\widetilde{Q}}\left[\sum_{k=2}^{K}(M \cdot \Delta t_k + A(t_k) \cdot p(t_k) \cdot \Delta t_k) \cdot e^{-\int_0^{t_k}(r(s)+p(s))ds}\right], \quad (1) \end{aligned}$$

where $r(t)$ is the (non-defaultable) short-rate process, $p(t)$ is the (annualised) prepayment intensity (prepayment speed) process and $\widetilde{Q}$ is the risk-neutral martingale measure. Formula (1) is a discrete version of the continuous time valuation formula in Goncharov (2005) which follows directly from similar results in the credit risk literature (see, e.g., Proposition 8.2.1 in Bielecki and Rutkowski (2002)). In the credit risk literature the process $p(t)$ is usually the default intensity. If the default intensity process can be set up within a Gaussian framework, a closed-form representation of formula (1) is, in general, possible. In many models and applications in practice the default intensity is modelled independently from $r(t)$. In the case of prepayment modelling, however, the prepayment intensity process $p(t)$ can not be assumed to be independent of the interest-rate process $r(t)$ since it is a well known fact that mortgage borrowers are more likely to refinance their loans and thus prepay their mortgages when interest rates decline. All non-refinancing related prepayment is usually labelled turnover or baseline prepayment. In the following we will decompose the overall prepayment intensity into the two independent components refinancing-related prepayment $p_{\text{refi}}(t)$ and baseline prepayment $p_0(t)$, i.e.

$$p(t) = p_{\text{refi}}(t) + p_0(t). \tag{2}$$

Following the argumentation in Kolbe and Zagst (2006) based on the Girsanov theorem for marked point processes, we introduce a multiplicative prepayment-risk adjustment parameter μ so that, under the risk-neutral measure $\widetilde{Q}$, the prepayment process has the dynamics

$$dp(t) = \mu \cdot (dp_{\text{refi}}(t) + dp_0(t)). \tag{3}$$

We will discuss the refinancing component and the turnover component of prepayment in the Subsections 2.1 and 2.2 before we finally put all components together for our closed-form formula in Subsection 2.3.

2.1 The short-rate model and the refinancing component

A crucial component of every MBS valuation model is an adequate model for the interest-rate term structure. For our closed-form formula we use a 1-factor Cox-Ingersoll-Ross (CIR) model as presented in Cox et al. (1985). In the CIR model, the risk-free short-rate dynamics under the risk-neutral measure $\widetilde{Q}$ are given by

$$dr(t) = (\theta_r - \hat{a}_r r(t))dt + \sigma_r \sqrt{r(t)} d\widetilde{W}_r(t), \tag{4}$$

where $\widetilde{W}_r$ is a $\widetilde{Q}$-Wiener process, $\hat{a}_r := a_r + \lambda_r \sigma_r^2$ with the market price of risk parameter λ_r and some positive constants θ_r, a_r, σ_r with $2\theta_r > \sigma_r^2$. The zero-coupon bond prices in the CIR model can be calculated analytically (see again Cox et al. (1985)) and are comprised in the following lemma:

Lemma 2.1. *In the CIR short-rate model and with* $r^c(t) := c \cdot r(t)$ *for some constant* $c \geq -\frac{\hat{a}_r^2}{2\sigma_r^2}$*, it holds that*

$$P^c(t,T) := E_{\widetilde{Q}}[e^{-\int_t^T r^c(s)ds} | \mathcal{F}_t] = e^{A^c(t,T) - B^c(t,T) r(t)} \tag{5}$$

where

$$\begin{aligned} B^c(t,T) &= c \cdot \frac{1 - e^{-\gamma^c (T-t)}}{\kappa_1 - \kappa_2 e^{-\gamma^c (T-t)}}, \\ A^c(t,T) &= \frac{2\theta_r}{\sigma_r^2} \log \left[\frac{\gamma^c e^{\kappa_2 \cdot (T-t)}}{\kappa_1 - \kappa_2 \cdot e^{-\gamma^c \cdot (T-t)}} \right] \end{aligned}$$

with $\gamma^c := \sqrt{\hat{a}_r^2 + 2\sigma_r^2 c}$*,* $\kappa_1 := \frac{\hat{a}_r}{2} + \frac{\gamma^c}{2}$ *and* $\kappa_2 := \frac{\hat{a}_r}{2} - \frac{\gamma^c}{2}$*.*

Proof. See Appendix. □

If $c = 1$ in Lemma 2.1, $P^c(t,T)$ is the price of a zero-coupon bond in the CIR model and we will write $P(t,T)$, γ, $B(t,T)$ and $A(t,T)$ instead of $P^c(t,T)$, γ^c, $B^c(t,T)$ and $A^c(t,T)$ in this case.

The refinancing incentive is usually modelled as a function of the spread between a security's weighted-average coupon (WAC) and current long-term interest rates which serve as a proxy for mortgage refinancing rates. While in some publications (e.g., Levin and Daras (1998) or Kolbe and Zagst (2006)) the 10yr par yield is used, we use the 10yr zero yield in this paper since this is a more convenient choice for our closed-form formula. Note that within the CIR framework the 10yr zero yield R_{10} is given by

$$R_{10}(t) = -a_{10} + b_{10} \cdot r(t), \tag{6}$$

where $a_{10} := \frac{A(t,t+10)}{10}$ and $b_{10} := \frac{B(t,t+10)}{10}$. Contrarily to Collin-Dufresne and Harding (1999) we do not use a purely linear functional form, but approximate an S-curve shape by defining

$$p_{\text{refi}}(t) = \beta \cdot \max(\min(\text{WAC} - R_{10}(t), \alpha), 0), \tag{7}$$

for some constant $\alpha > 0$, which results in a spread-refinancing prepayment relationship as shown in Figure 1. This functional form offers two major advantages compared to a purely linear functional form:

- The S-like relationship between the spread and the refinancing-driven prepayment, which has been confirmed empirically by, e.g., Levin and Daras (1998), is accounted for.
- Refinancing-driven prepayment can never become negative.

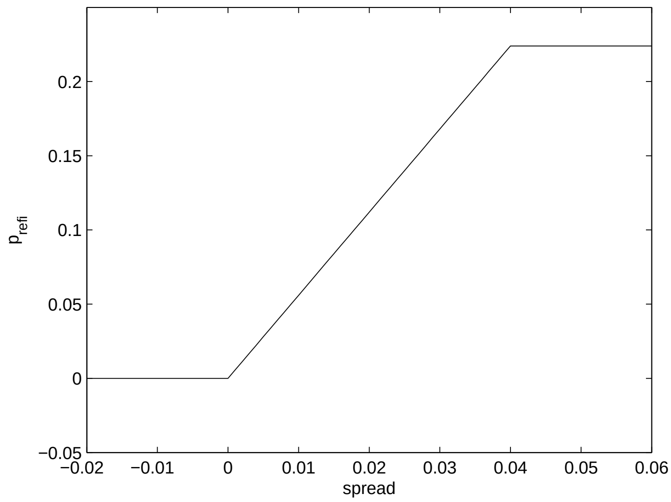

Figure 1: Assumed functional form of the relationship between the contract rate spread (i.e. the spread between the WAC and the current 10yr treasury zero rate) and the refinancing-related (annualised) prepayment speed. The parameter β in (7) is set to 5.6 as estimated in Section 3 and α is set to 0.04.

Using (6) and noting that for some constants $a, b, c \in \mathbb{R}$, $b > c$, we have

$$\max(\min(a - x, b), c) = a - x + \max(x - (a - c), 0) - \max(a - b - x, 0),$$

formula (7) gets:

$$\begin{aligned} p_{\text{refi}}(t) &= \beta \cdot \text{WAC} + \beta a_{10} - \beta b_{10} r(t) \\ &\quad + \beta b_{10} \cdot \max\left(r(t) - \frac{\text{WAC} + a_{10}}{b_{10}}, 0 \right) \\ &\quad - \beta b_{10} \cdot \max\left(\frac{\text{WAC} + a_{10} - \alpha}{b_{10}} - r(t), 0 \right). \end{aligned} \tag{8}$$

Now, consider the term $E_{\widetilde{Q}}\left[e^{-\int_0^{t_k}(r(s)+p_{\text{refi}}(s))ds}\right]$. Defining

$$\widetilde{r}(t) := r^{(1-\beta b_{10})}(t) = (1 - \beta b_{10}) \cdot r(t)$$

we get by using (8):

$$\begin{aligned} E_{\widetilde{Q}}\left[e^{-\int_0^{t_k}(r(s)+p_{\text{refi}}(s))ds}\right] &= E_{\widetilde{Q}}\Big[e^{-\int_0^{t_k} \beta \cdot \text{WAC} + \beta a_{10} + \widetilde{r}(s) ds} \cdot \\ &\quad e^{-\int_0^{t_k} \beta b_{10} \cdot \max\left(r(s) - \frac{\text{WAC}+a_{10}}{b_{10}}, 0\right) ds} \cdot \\ &\quad e^{\int_0^{t_k} \beta b_{10} \cdot \max\left(\frac{\text{WAC}+a_{10}-\alpha}{b_{10}} - r(s), 0\right) ds}\Big]. \end{aligned} \tag{9}$$

Theorem 2.2. *Within the model setting as previously introduced and defining*

$$C(t_k) := e^{-t_k \cdot (\beta \cdot WAC + \beta a_{10})}$$

the expression

$$P^{refi}(0, t_k) := E_{\widetilde{Q}}\left[e^{-\int_0^{t_k}(r(s)+p_{\text{refi}}(s))ds}\right]$$

can be reasonably approximated in the following way:

$$\begin{aligned} P^{refi}(0, t_k) &\approx C(t_k) \cdot \widetilde{P}(0, t_k) - C(t_k) \cdot \beta b_{10} \cdot \widetilde{\text{Cap}}(r, 0, t_k, r_{Cap}, \Delta t) \\ &\quad + C(t_k) \cdot \beta b_{10} \cdot \widetilde{\text{Floor}}(r, 0, t_k, r_{Floor}, \Delta t), \end{aligned} \tag{10}$$

where, according to Lemma 2.1,

$$\begin{aligned} \widetilde{P}(0, t_k) &:= P^{(1-\beta b_{10})}(0, t_k) \\ r_{Cap} &:= \frac{WAC + a_{10}}{b_{10}} \\ r_{Floor} &:= \frac{WAC + a_{10} - \alpha}{b_{10}} \\ \widetilde{\text{Cap}}(r, 0, T, r_X, \Delta t) &:= \sum_{k=1}^{T/\Delta t} \Delta t \cdot \left[\frac{q + 1 + u_k}{c_k} - \frac{u_k}{c_k} \cdot \chi^2(2c_k r_X, 2q + 6, 2u_k)\right. \\ &\quad \left. - \frac{q+1}{c_k} \cdot \chi^2(2c_k r_x, 2q + 4, 2u_k) - r_X + r_X \cdot \chi^2(2c_k r_X, 2q + 2, 2u_k)\right] \\ \widetilde{\text{Floor}}(r, 0, T, r_X, \Delta t) &:= \sum_{k=1}^{T/\Delta t} \Delta t \cdot \left[r_X \cdot \chi^2(2c_k r_X, 2q + 2, 2u_k)\right. \\ &\quad \left. - \frac{u_k}{c_k} \cdot \chi^2(2c_k r_X, 2q + 6, 2u_k) - \frac{q+1}{c_k} \cdot \chi^2(2c_k r_X, 2q + 4, 2u_k)\right] \\ c_k &:= \frac{2\hat{a}_r}{\sigma_r^2 \cdot (1 - e^{-\hat{a}_r \cdot k \cdot \Delta t})} \\ u_k &:= c_k \cdot r(0) \cdot e^{-\hat{a}_r \cdot k \cdot \Delta t} \\ q &:= \frac{2\theta_r}{\sigma_r^2} - 1 \end{aligned}$$

and $\chi^2(\cdot; a, b)$ denotes the cdf of the non-central Chi-square distribution with degrees of freedom parameter a and non-centrality parameter b.

Proof. See Appendix. □

Note that the notation "$\widetilde{\text{Cap}}$" has not been chosen without motive. If one equates the linear interest rate at time t for the period from t to $t + \Delta t$ with the short rate $r(t)$, the expression

$$\max\left(r(k \cdot \Delta t) - \frac{(\text{WAC} + a_{10})}{b_{10}}, 0\right) \cdot \Delta t$$

in (9) is simply the payoff of a standard caplet from $k \cdot \Delta t$ to $(k+1) \cdot \Delta t$ with cap rate $\underbrace{r_{Cap}} := (\text{WAC} + a_{10})/b_{10}$. A similar consideration applies for the notation "Floor". Note also that the accuracy of the approximation in Theorem 2.2 depends, of course, on the parameter values and on the WAC of the security. In general, the approximation should be better for MBS around the current-coupon level than for deep discounts (low-coupon securities, traded below par) or very high premiums (high-coupon securities, traded above par). We typically have $\Delta t_k = 1/12$ (i.e. 1 month) for all $k = 1, ..., K$. Hence, $\Delta t = 1/12$ is a natural choice for the interval length of the discretisation in (10).

2.2 The baseline prepayment

We model the baseline or turnover component of prepayment within a two-factor Gaussian process framework where both factors follow Vasicek processes. The second factor is fit to the GDP growth in the US, accounting for the dependence between general economic conditions and turnover prepayment. Of course, any other observable factor, e.g. a suitable house price index, could be used instead of or in addition to the GDP growth factor. While our empirical results have turned out to be satisfactory with the GDP growth as second factor in the baseline prepayment model (see also Sections 3 and 4), house prices have been used for example by Kariya et al. (2002), Sharp et al. (2006) or Downing et al. (2005). Our baseline prepayment processes are given by their $\widetilde{Q}$-dynamics

$$\begin{aligned} dp_0(t) &= (\theta_p + b_{pw} w(t) - \hat{a}_p p_0(t))dt + \sigma_p d\widetilde{W}_p(t), \qquad (11) \\ dw(t) &= (\theta_w - \hat{a}_w w(t))dt + \sigma_w d\widetilde{W}_w(t), \end{aligned}$$

where $\widetilde{W}_p, \widetilde{W}_w$ are independent $\widetilde{Q}$-Wiener processes (independent of the previously defined $\widetilde{W}_r$) and $\hat{a}_i := a_i + \lambda_i \sigma_i^2$, $i = p, w$, for the two prepayment-risk-adjustment parameters λ_p, λ_w.

In order to be able to calculate (1) we have to evaluate the expression

$$\begin{aligned} \widetilde{P}^d(t,T) &:= E_{\widetilde{Q}}[e^{-\int_t^T (\widetilde{r}(s)+p_0(s))ds}] \\ &= E_{\widetilde{Q}}[e^{-\int_t^T \widetilde{r}(s)ds}] \cdot E_{\widetilde{Q}}[e^{-\int_t^T p_0(s)ds}] \\ &=: \widetilde{P}(t,T) \cdot P^{base}(t,T), \end{aligned}$$

where $\widetilde{r}(t) := r^{(1-\beta b_10)}(t)$, $\widetilde{P}(t,T) := P^{(1-\beta b_10)}(t,T)$. The letter d in the superscript of $\widetilde{P}^d(t,T)$ is used in analogy to the reduced-form credit risk literature.

Theorem 2.3. *In the model set-up as previously introduced it holds that*

$$P^{base}(t,T) = e^{A^d(t,T) - C^d(t,T)p_0(t) - D^d(t,T)w(t)}$$

with

$$
\begin{aligned}
C^d(t,T) &= \frac{1}{\hat{a}_p}\left(1-e^{-\hat{a}_p(T-t)}\right), \\
D^d(t,T) &= \frac{b_{pw}}{\hat{a}_p}\left(\frac{1-e^{-\hat{a}_w(T-t)}}{\hat{a}_w}+\frac{e^{-\hat{a}_w(T-t)}-e^{-\hat{a}_p(T-t)}}{\hat{a}_w-\hat{a}_p}\right), \\
A^d(t,T) &= \int_t^T \frac{1}{2}\left(\sigma_p^2 C^d(l,T)^2+\sigma_w^2 D^d(l,T)^2\right) \\
&\quad -\theta_p C^d(l,T)-\theta_w D^d(l,T)dl.
\end{aligned}
$$

Proof. See Appendix. □

Note, that we have associated the prepayment speed $p(t)$ with the intensity of prepayment. In our model specification, however, $p(t)$ can have negative values, albeit, in general, with only small probabilities. Prepayments for ordinary fixed-rate MBS can, of course, never be negative. Furthermore the association of the process $p(t)$ with the prepayment intensity (and likewise the association of the baseline prepayment process $p_0(t)$ with the corresponding default intensity process in the credit risk literature) is not unproblematic from a technical point of view for the same reason: intensities can never be negative. We thus consider the processes $p(t)$ and $p_0(t)$ as proxies for the respective intensity processes. The fact that both $p(t)$ and $p_0(t)$ are negative only with small probabilities justifies this approach (see also Schönbucher (2003), p. 167, for a further discussion of this topic).

2.3 The closed-form approximation

With the ingredients developed in the previous subsections the expressions

- $E_{\widetilde{Q}}\left[M\cdot e^{-\int_0^{t_1} r(s)ds}\right]=M\cdot P(0,t_1)$
- $E_{\widetilde{Q}}\left[M\cdot e^{-\int_0^{t_k}(r(s)+p(s))ds}\right]=M\cdot P^{refi}(0,t_k)\cdot P^{base}(0,t_k)$

in (1) can readily be evaluated for all k. This is not yet the case for the terms involving $p(t_k)$ as a factor.

Lemma 2.4. *In the previously introduced model set-up it holds that*

$$
\begin{aligned}
E_{\widetilde{Q}}\left[p_0(t_k)\cdot e^{-\int_0^{t_k}(r(s)+p(s))ds}\right] \approx\ & C(t_k)\cdot \widetilde{P}^d(0,t_k)\cdot \tilde{f}^d(0,t_k) \\
&-C(t_k)\cdot \beta b_{10}\cdot P^{base}(0,t_k)\cdot \tilde{f}^d(0,t_k)\cdot \widetilde{\mathrm{Cap}}(r,0,t_k,r_{Cap},\Delta t_k) \\
&+C(t_k)\cdot \beta b_{10}\cdot P^{base}(0,t_k)\cdot \tilde{f}^d(0,t_k)\cdot \widetilde{\mathrm{Floor}}(r,0,t_k,r_{Floor},\Delta t_k),
\end{aligned}
$$

where $\tilde{f}^d(0,t_k)$ *is the "baseline spread forward rate", i.e.*

$$\tilde{f}^d(0,t_k) = -\frac{\partial}{\partial t_k}\ln P^{base}(0,t_k).$$

Proof. As a first step, recall the well-known result (see, e.g., Schmid (2004), p. 243) saying that

$$E_{\widetilde{Q}}\left[e^{-\int_0^T r(l)dl} r(T)|\mathcal{F}_0\right] = -E_{\widetilde{Q}}\left[e^{-\int_0^T r(l)dl}|\mathcal{F}_0\right]\cdot\frac{\partial}{\partial T}\ln P(0,T). \qquad (12)$$

Now, if we use use the independence between $(r(t),p_{\text{refi}}(t))$ and $p_0(t)$, apply (12) to

$$\begin{aligned} E_{\widetilde{Q}}\left[p_0(t_k)\cdot e^{-\int_0^{t_k}(r(s)+p(s))ds}\right] &= E_{\widetilde{Q}}\left[e^{-\int_0^{t_k}(r(s)+p_{\text{refi}}(s))ds}\right]\cdot \\ &\quad E_{\widetilde{Q}}\left[p_0(t_k)\cdot e^{-\int_0^{t_k}p_0(s)ds}\right], \end{aligned}$$

the lemma follows dircectly if we recall (10). □

This leaves us with the term $E_{\widetilde{Q}}\left[p_{\text{refi}}(t_k)\cdot e^{-\int_0^{t_k}(r(s)+p(s))ds}\right]$.

Lemma 2.5. *Within the previously introduced model set-up it holds that:*

$$\begin{aligned} E_{\widetilde{Q}}\left[p_{refi}(t_k)\cdot e^{-\int_0^{t_k}(r(s)+p(s))ds}\right] &= -P^{base}(0,t_k)\cdot P^{refi}(0,t_k) \\ &\quad \cdot\frac{\partial}{\partial t_k}\ln\left[\frac{P^{refi}(0,t_k)}{P(0,t_k)}\right]. \end{aligned} \qquad (13)$$

Proof. If we define the t_k-forward measure Q^{t_k} in the usual way via its Radon-Nikodym derivative $L(T)$ with respect to $\widetilde{Q}$ by

$$L(t) = \frac{dQ^{t_k}}{d\widetilde{Q}}\bigg|\mathcal{F}_t = \frac{P(t,t_k)}{P(0,t_k)\cdot e^{\int_0^{t_k}r(s)ds}}$$

for $t\in[0,t_k]$ and use (12) we obtain:

$$\begin{aligned} E_{\widetilde{Q}}\left[p_{\text{refi}}(t_k)\cdot e^{-\int_0^{t_k}(r(s)+p(s))ds}\right] &= E_{\widetilde{Q}}\left[p_{\text{refi}}(t_k)\cdot e^{-\int_0^{t_k}(r(s)+p_{\text{refi}}(s))ds}\right]\cdot \\ &\quad P^{base}(0,t_k) \\ &= P(0,t_k)\cdot E_{Q^{t_k}}\left[p_{\text{refi}}(t_k)\cdot e^{-\int_0^{t_k}p_{\text{refi}}(s)ds}\right]\cdot P^{base}(0,t_k) \\ &= -P(0,t_k)\cdot E_{Q^{t_k}}\left[e^{-\int_0^{t_k}p_{\text{refi}}(s)ds}\right]\cdot \\ &\quad \frac{\partial}{\partial t_k}\ln E_{Q^{t_k}}\left[e^{-\int_0^{t_k}p_{\text{refi}}(s)ds}\right]\cdot P^{base}(0,t_k) \\ &= -P^{refi}(0,t_k)\cdot\frac{\partial}{\partial t_k}\ln\left[\frac{P^{refi}(0,t_k)}{P(0,t_k)}\right]\cdot P^{base}(0,t_k) \end{aligned}$$

□

Note that by using Theorem 2.3 and the approximation given in Theorem 2.2, it is straightforward to evaluate the terms in (13).

We can finally summarize our results in the following theorem.

Theorem 2.6. *Within the model specification as previously introduced the value $V(0)$ of a fixed-rate mortgage-backed security with K outstanding payment dates at time 0 is approximately given by:*

$$V(0) \approx S_1 + S_2 + S_3 - \Delta_1 + \Delta_2 \tag{14}$$

with

$$\begin{aligned}
S_1 &= M \cdot \Delta t_1 \cdot P(0,t_1) + \sum_{k=2}^{K} M \cdot \Delta t_k \cdot C(t_k) \cdot \widetilde{P}^d(0,t_k) \\
S_2 &= \sum_{k=1}^{K} C(t_k) \cdot \widetilde{P}^d(0,t_k) \cdot A(t_k) \cdot \Delta t_k \cdot \tilde{f}^d(0,t_k) \\
S_3 &= -\sum_{k=1}^{K} C(t_k) \cdot \widetilde{P}^d(0,t_k) \cdot A(t_k) \cdot \Delta t_k \cdot \\
&\quad \frac{\partial}{\partial t_k} \ln\left[\frac{P^{refi}(0,t_k)}{P(0,t_k)}\right]
\end{aligned}$$

and

$$\begin{aligned}
\Delta_1 &= \sum_{k=1}^{K} \widetilde{\mathrm{Cap}}(r,0,t_k,r_{Cap},\Delta t_k) \cdot C(t_k) \cdot P^{base}(0,t_k) \cdot \\
&\quad \beta b_{10} \cdot \Bigg[M \cdot \Delta t_k + A(t_k) \cdot \Delta t_k \cdot \tilde{f}^d(0,t_k) - A(t_k) \cdot \Delta t_k \cdot \\
&\quad \frac{\partial}{\partial t_k} \ln\left[\frac{P^{refi}(0,t_k)}{P(0,t_k)}\right]\Bigg] \\
\Delta_2 &= \sum_{k=1}^{K} \widetilde{\mathrm{Floor}}(r,0,t_k,r_{Floor},\Delta t_k) \cdot C(t_k) \cdot P^{base}(0,t_k) \cdot \\
&\quad \beta b_{10} \cdot \Bigg[M \cdot \Delta t_k + A(t_k) \cdot \Delta t_k \cdot \tilde{f}^d(0,t_k) - A(t_k) \cdot \Delta t_k \cdot \\
&\quad \frac{\partial}{\partial t_k} \ln\left[\frac{P^{refi}(0,t_k)}{P(0,t_k)}\right]\Bigg]
\end{aligned}$$

Formula (14) can readily be evaluated once the model parameters have been estimated and calibrated (see Section 3). From Theorem 2.6 it is also easy to see how the most common mortgage derivatives, i.e. Interest-Only (IO) and

Principal-Only (PO) securities, can be priced within our modelling framework. If we split up the mortgage payment M into the interest payment M^I and regular principal repayment M^P, so that $M{=}M^I + M^P$, and denote

$$\begin{aligned}
S_1^I &:= M^I \cdot \Delta t_1 \cdot P(0,t_1) + \sum_{k=2}^{K} M^I \cdot \Delta t_k \cdot C(t_k) \cdot \widetilde{P}^d(0,t_k), \\
\Delta_1^I &:= \sum_{k=1}^{K} \widetilde{\mathrm{Cap}}(r,0,t_k,r_{Cap},\Delta t_k) \cdot C(t_k) \cdot P^{base}(0,t_k) \cdot \beta b_{10} \cdot M^I \cdot \Delta t_k \\
\Delta_2^I &:= \sum_{k=1}^{K} \widetilde{\mathrm{Floor}}(r,0,t_k,r_{Floor},\Delta t_k) \cdot C(t_k) \cdot P^{base}(0,t_k) \cdot \beta b_{10} \cdot M^I \cdot \Delta t_k \\
S_1^P &:= S_1 - S_1^I \\
\Delta_1^P &:= \Delta_1 - \Delta_1^I \\
\Delta_2^P &:= \Delta_2 - \Delta_2^I,
\end{aligned}$$

we obtain the following two corollaries, which conclude this section.

Corollary 2.7. *The value $V_{IO}(0)$ of an Interest-Only security with K outstanding payment dates at time 0 is given by:*

$$V_{IO}(0) = S_1^I - \Delta_1^I + \Delta_2^I$$

Corollary 2.8. *The value $V_{PO}(0)$ of a Principal-Only security with K outstanding payment dates at time 0 is given by:*

$$V_{PO}(0) = S_1^P + S_2 + S_3 - \Delta_1^P + \Delta_2^P$$

3 Parameter estimation and model calibration

The available data for this study consists of US treasury strip par rates and monthly historic prepayment data for large issues of 30yr fixed-rate mortgage-backed securities of the GNMA I and GNMA II programs. In addition to this we have monthly historic prices of generic GNMA 30yr pass-through MBS for a wide range of different coupons as traded on a to-be-announced (TBA) basis. All data were obtained from Bloomberg.

Weekly US treasury strip zero rates, obtained from the par rates by standard bootstrapping, from 1993 to 2005 are used for the estimation of the parameters of the CIR interest-rate model. We estimate the parameters with a state-space approach which integrates time-series information of different maturities, similar to the approach described in Geyer and Pichler (1999). Estimation of the unobservable state variables (i.e. the short-rate) is done with an approximative Kalman filter where the transition densities are supposed to be normal. For the maximisation of the log-likelihood we use a combined Downhill Simplex/Simulated Annealing algorithm as described in Press et al. (1992), which we also use for all following maximisation and optimisation steps.

The parameters θ_w, a_w, σ_w of the (real-world) GDP growth process, i.e. the parameters of the second factor of our baseline prepayment process, are estimated by simple Maximum Likelihood. We use quarterly growth data and obtain monthly values by cubic spline interpolation. The parameters $\theta_p, a_p, \sigma_p, b_{pw}$ are estimated by a Kalman filter for state space models with the efficient numerical algorithms as described in Koopman et al. (1999). The measurement equation of the state space model is given by (2) with the historically observed annualised prepayment rates $p(t_k)$ and p_{refi} as specified in (7). We use the historic pool data of a total of $N = 8$ individual mortgage pools for the empirical prepayment model, so that we obtain the measurement equation:

$$\begin{pmatrix} p_1(t_k) \\ \vdots \\ p_N(t_k) \end{pmatrix} = \begin{pmatrix} p_{1,\text{refi}}(t_k) \\ \vdots \\ p_{N,\text{refi}}(t_k) \end{pmatrix} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \cdot p_0(t_k) + \boldsymbol{\epsilon}_k, \tag{15}$$

where we assume that $\boldsymbol{\epsilon}_k \sim N_N\left(\mathbf{0}, h_p^2 \cdot \mathbf{I}_N\right)$. The corresponding MBS were issued between 1993 and 1996 with more than USD 50m of residential mortgage loans in each of the eight pools and have coupons between 6% and 9%, so that both discounts and premiums are included in our sample. Since in 2002-2004 prepayment speeds were very high, we only use the data until 2004 for parameter estimation in the prepayment model in order to avoid noise in our observations caused by small pool sizes after 2004. The transition equation for the (unobservable) baseline prepayment process can be obtained from (11). For stability reasons, we use $w(t)$ as an external input to the model and obtain the transition equation

$$p_0(t_{k+1}) = e^{-a_p \Delta t_{k+1}} \cdot p_0(t_k) + \frac{\theta_p + b_{pw} w(t_k)}{a_p} \cdot (1 - e^{-a_p \Delta t_{k+1}}) + \eta_{k+1}$$

with

$$\eta_{k+1} \sim N_1\left(0, \frac{\sigma_p^2}{2a_p}(1 - e^{-2a_p \Delta t_{k+1}})\right).$$

The estimates of the interest-rate model parameters and of the (real-world) prepayment model parameters are given in Table 1. The standard errors are estimates obtained from a moving block bootstrap procedure (see, e.g. Lahiri (2003) for details on block bootstrapping techniques).

In the next step, we turn our attention to the prepayment-risk adjustment parameters $\mu, \lambda_p, \lambda_w$. By simply setting $\mu = 1$ and $\lambda_p = \lambda_w = 0$ we can conduct a classical OAS analysis since in this case the prepayment speed

	Parameter	Estimate	(Std. error)
Short-rate process	θ_r	0.014	(0.0056)
	a_r	0.41	(0.12)
	σ_r	0.059	(0.0073)
	$\hat{a}_r$	0.20	(0.10)
	h_r	0.0044	($4.8 \cdot 10^{-4}$)
GDP growth process	θ_w	0.019	(0.0099)
	a_w	1.43	(0.79)
	σ_w	0.002	($4.3 \cdot 10^{-4}$)
Baseline prepayment process	θ_p	0.43	(0.20)
	a_p	0.75	(0.56)
	σ_p	0.12	(0.057)
	b_{pw}	-22.6	(5.03)
	h_p	0.085	(0.012)
Regression parameter	β	5.6	(0.90)

Table 1: Estimates of the interest-rate model and real-world prepayment model parameters where h_r and h_p are the measurement std. errors of the respective state space models.

enters with its real-world dynamics into the overall model and the OAS is needed to equate model prices to actually observed market prices. Yet, in this paper we are primarily interested in a prepayment-risk-neutral valuation following the argumentation in Levin and Davidson (2005) and Kolbe and Zagst (2006). Using price data of different coupon levels of GNMA TBA pass-through securities we calibrate the prepayment-risk adjustment parameters in such a way that the Euclidean norm of the vector of differences between the market prices and model prices of the securities on a particular sample day is minimised. GNMA securities which are traded on a TBA basis are good benchmark securities for the calibration exercise since they are highly liquid. We thus do not have to worry about liquidity effects/premia. In this study we consider monthly price data of five generic GNMA TBA pass-throughs with coupons between 6% and 8% from 1996 to 2006. We recalibrate the risk-adjustment parameters once a year in October in order to account for changing perceptions of prepayment risk over time.

At this point we would like to explain how the prepayment-risk adjustment parameters are able to account for the two distinct types of prepayment risk. The fact that there are two distinct types of prepayment risk, refinancing understatement and turnover overstatement, was already mentioned in Levin and Davidson (2005) and discussed in detail in Kolbe and Zagst (2006). On the one hand, an investor in discounts experiences losses if the turnover component is overestimated and pure turnover-related prepayment is slower than expected. In this case the average life of the security is extended, decreasing the cash flow stream's present value. On the other hand, the refinancing component is the major concern of an investor in premiums since the average life of premiums decreases if refinancing-related prepayment is faster than originally estimated. This would evidently result in a loss for the holder of a premium MBS. For $\mu > 1$ both, refinancing and baseline prepayment, is accelerated under the risk-neutral pricing measure, compared to the real-world measure. The parameters λ_p and λ_w, however, only affect the baseline prepayment. The higher λ_p the slower the expected prepayment rates under the risk-neutral pricing measure. For the estimates calibrated to the data of Oct-1996 we obtain $\mu = 1.28$, $\lambda_p = 10.0$ and $\lambda_w = -165$. For these estimates Figure 2 shows the expecetd prepayment rates under the risk-neutral pricing measure as a function of the spread variable. Compared to the real-world measure, higher prepayment rates are expected under the risk-neutral pricing measure in the premium area (i.e. for high spread values), while slower prepayment rates are expected in the discount area (i.e. for low values of the spread variable). In some sense, the

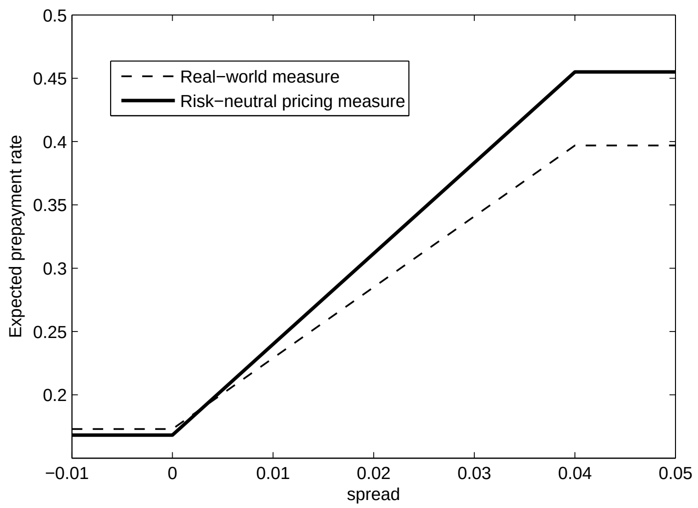

Figure 2: Expected prepayment rates under the real-world measure and under the risk-neutral pricing measure as a function of the spread variable with the prepayment-risk adjustment parameters calibrated to GNMA market prices of Oct-1996.

expected prepayment speeds under the risk-neutral pricing measure could be considered as "implied expected prepayment rates", implied by MBS market prices. Figure 3 shows how these "implied expected prepayment rates" evolve over time when we re-estimate the prepayment-risk adjustment parameters once a year. The parameter μ varies around its mean 1.24 (with a standard deviation of 0.21), the parameter λ_p around 16.9 (std. dev. 18.0) and the parameter λ_w around -84 (std. dev. 68).

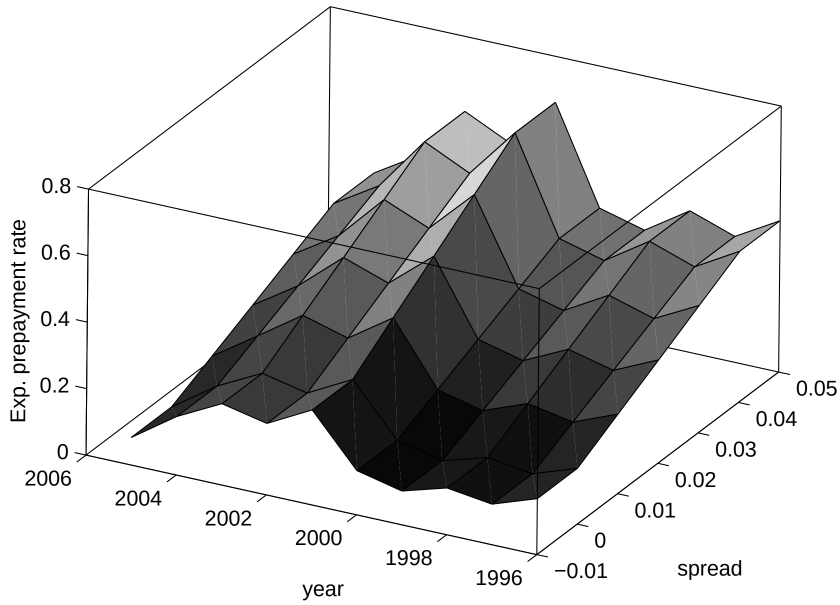

Figure 3: Expected prepayment speeds under the risk-neutral pricing measure from 1996 to 2006 as a function of the spread variable when the prepayment-risk adjustment parameters are recalibrated once a year.

4 Empirical results

The main contribution of this paper, as previously mentioned, is to provide a closed-form (and thus computationally very efficient) approximation of the value of fixed-rate mortgage-backed securities. This is particularly useful for risk and portfolio management purposes where other valuation methods may not be feasible due to their computational burden. Yet, a closed-form approximation of the securities' values is only useful if the model is able to track major price movements of actually traded securities. In order to assess our model's performance and adequacy empirically, we use the price data of the GNMA TBA pass-throughs from 1996 to 2006. In order to simplify the analysis we assume that each MBS was issued 6 months before the valuation month. Since in this paper we do not model loan age effects anyway, this is not a major restriction. In a first step, we price the securities with the risk-adjustment parameters recalibrated once a year, as described in the previous section. Figure 4 shows that our model is able to track the price movements of the MBS successfully across the whole range of coupons, including both discounts and premiums. The average absolute pricing error

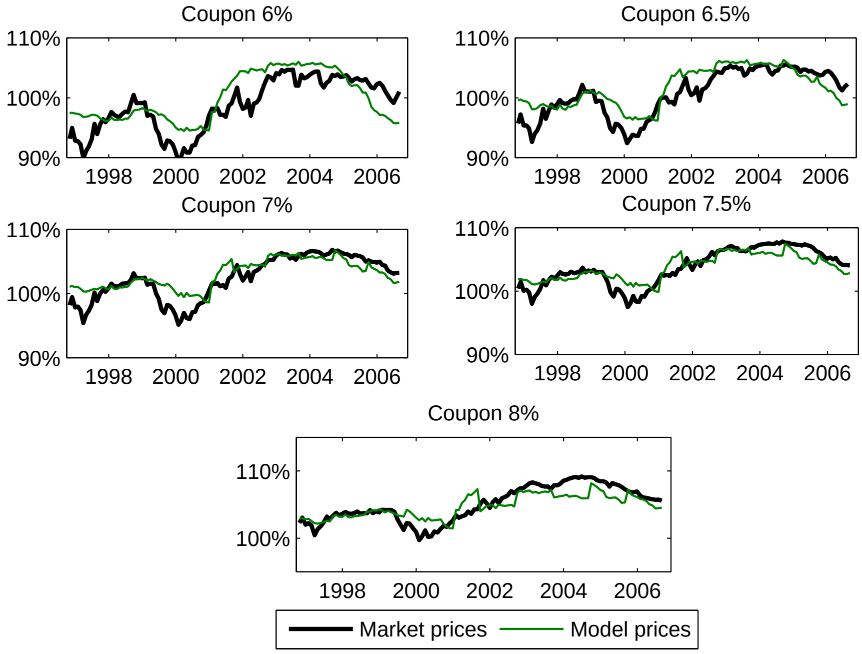

Figure 4: Market and model prices for a series of generic GNMA TBA pass-throughs (Bloomberg ticker GNSF) with different coupons from 1996 to 2006 when the prepayment-risk adjustment parameters are recalibrated once a year.

for each coupon is shown in Table 2. The overall average absolute pricing error over the entire sample is 159 basis points, i.e. just above 1.5%.

In the same empirical setting we also want to discuss the value of our piecewise linear approximation of the refinancing S-curve compared to a purely linear functional form like in the model developed by Collin-Dufresne and Harding (1999). Since the Collin-Dufresne/Harding model is based on an option-theoretic approach, it is hard to compare their model with ours directly. In the Collin-Dufresne/Harding model, the refinancing (annualised) prepayment speed for a fixed-rate MBS with maturity T is given by

$$p_{\text{refi}}(t) = a_0 + a_1 \cdot \ln \frac{P(0,T)}{P(t,T)}$$

for some constants a_0, a_1. Thus, the spread explanatory variable is defined in a slightly different way compared to our model. This difference, as well as the fact that Collin-Dufresne and Harding (1999) use a Vasicek process for the short-rate, can be considered as minor differences between the models. Apart from the restriction to one stochastic factor, the major restriction in the Collin-Dufresne/Harding model is the purely linear form for the approximation of the refinancing S-curve. Within our model framework, we want to test empirically whether the piecewise linear approximation presented in this paper does add explanatory power to the pricing model. For this purpose we re-estimate our model with a purely linear functional form. I.e., instead of (7) we set:

$$p_{\text{refi}}(t) = \beta \cdot (\text{WAC} - R_{10}(t)).$$

Note that there is no need for an intercept here since we still incorporate the baseline prepayment process $p_0(t)$. This, of course, makes the formula much easier since we do not have to deal with the rather complex formulas of Theorem 2.2. We also re-calibrate the risk-adjustment parameters once a year and price the five different coupon securities with this model from 1996 to 2006. The results as shown in Table 2 indicate that the piecewise linear approximation yields indeed better results than the purely linear functional form, almost across the whole coupon range. The overall average absolute pricing error is 266 basis points in the model with a purely linear functional form, compared to 159 basis points in our full model.

Coupon	Average abs. pricing error (full model, in basis points)	Average abs. pricing error (linear model, in basis points)
6%	266	380
6.5%	187	142
7%	141	165
7.5%	116	230
8%	121	402

Table 2: Average absolute pricing errors of our full model and of the model with a purely linear spread/prepayment speed relation for a series of generic GNMA TBA pass-throughs (Bloomberg ticker GNSF) with different coupons from 1996 to 2006 when the prepayment-risk adjustment parameters are recalibrated once a year.

In a second step we calibrate the prepayment-risk adjustment parameters only once to the data of Oct-1996 and leave them unchanged for the rest of the time period considered in this study. In this out-of sample experiment (w.r.t. the prepayment-risk adjustment parameters), our model is still able to explain the observed market prices with a satisfactory accuracy. The average absolute pricing error for each coupon is shown in Table 3 together with the results obtained by the model with a purely linear functional form. These results provide further evidence for the value of our piecewise linear approximation of the refinancing S-curve. The overall average absolute pricing error in this out-of-sample setting is 303 basis points in the model with a purely linear functional form, compared to 195 basis points in our full model.

Coupon	Average abs. pricing error (full model, in basis points)	Average abs. pricing error (linear model, in basis points)
6%	311	300
6.5%	248	207
7%	183	218
7.5%	146	293
8%	124	498

Table 3: Out-of-sample average absolute pricing errors of our full model and of the model with a purely linear spread/prepayment speed relation for a series of generic GNMA TBA pass-throughs (Bloomberg ticker GNSF) with different coupons from 1996 to 2006.

5 Conclusion

In this paper we have presented a closed-form formula which approximates the value of fixed-rate mortgage-backed securities based on reduced-form techniques adapted from the credit risk literature. Since the closed-form formula is computationally highly efficient and reduces the computational burden of MBS valuation drastically, our approach may be particularly useful for risk and portfolio management purposes where portfolios of MBS have to be revaluated frequently. So far (to the authors' best knowledge), the paper by Collin-Dufresne and Harding (1999) is the only previous paper which explicitly develops a closed-form formula for MBS pricing (based on an option-theoretic approach). Yet, compared to this previous approach, our model offers two major extensions. First, we are not restricted to a single stochastic factor. In addition to our 1-factor CIR interest-rate model we introduce two additional stochastic factors to model the baseline prepayment. The second factor of the baseline prepayment model is fit to the quarterly GDP growth rate in the US which allows us to account for the dependence between general economic conditions and turnover-related prepayment. Second, we do not approximate the usual S-like relation between coupon spread and refinancing-related prepayment by a purely linear functional form but introduce a piecewise linear approximation. Our results indicate that this contributes to a significant improvement of the model performance.

Applied to historic price data of 30yr GNMA pass-throughs traded on a TBA basis, our model proves to be able to explain major market price movements succesfully for a wide range of coupons. The overall average absolute pricing error of 159 basis points in our sample (with a yearly recalibration of prepayment-risk adjustment parameters) indicates a highly satisfactory accuracy of our model for risk and portfolio management purposes.

Our model also provides a further motivation for a thorough empirical study about which and how many factors are needed to adequately price mortgage-backed securities. In many previous pricing models this discussion was restricted due to the model specification and the computational burden associated with additional factors. While this is beyond the scope of this paper, our model may provide a basis for further empirical contributions in this field.

A Proof of Lemma 2.1:

For $c = 1$ we have the well-known formulas for zero-coupon bond prices in the CIR model. For $c \geq 0$ in general, we get the dynamics of $r^c(t)$ under $\widetilde{Q}$ by a simple application of the Ito-formula and obtain:

$$r^c(t) = (\theta_r^c - \hat{a}_r r^c(t))dt + \sigma_r^c \sqrt{r^c(t)} d\widetilde{W}_r(t)$$

with

$$\begin{aligned} \theta_r^c &= c \cdot \theta_r, \\ \sigma_r^c &= \sqrt{c} \cdot \sigma_r \end{aligned}$$

and the statement follows directly from, e.g., Zagst (2002), p.126/127. For $-\frac{\hat{a}_r^2}{2\sigma_r^2} \leq c < 0$, however, the result is less straightforward. We therefore explicitly give the detailed proof in the following.

From the Feynman-Kac representation of the Cauchy-Problem (see, e.g. Zagst (2002), [2.6]) we know that $P^c(t,T)$ must satisfy:

$$P_t^c + (\theta_r - \hat{a}_r r)P_r^c + \frac{1}{2} \cdot \sigma_r^2 \cdot r P_{rr}^c = c \cdot r \cdot P^c \tag{16}$$

with boundary condition $P^c(T,T) = 1$. Since

$$\begin{aligned} P_r^c &= -B^c \cdot P^c, \\ P_t^c &= P^c \cdot (A_t^c - r \cdot B_t^c), \\ P_{rr}^c &= (B^c)^2 \cdot P^c, \end{aligned}$$

it follows from (16) that

$$A_t^c(t,T) - \theta_r B^c(t,T) - r \cdot (c - \frac{1}{2} \cdot \sigma_r^2 \cdot (B^c(t,T))^2 + B_t^c(t,T) - \hat{a}_r \cdot B^c(t,T)) = 0$$

with $A^c(T,T) = B^c(T,T) = 0$. This leads to the Riccati-style equations

$$c - \frac{1}{2} \cdot \sigma_r^2 \cdot (B^c(t,T))^2 + B_t^c(t,T) - \hat{a}_r B^c(t,T) = 0$$

with $B^c(T,T) = 0$ and

$$A_t^c(t,T) = \theta_r B^c(t,T)$$

with $A^c(T,T) = 0$. By simple calculations it can be verified that

$$\begin{aligned} B^c(t,T) &= c \cdot \frac{1 - e^{-\gamma^c (T-t)}}{\kappa_1 - \kappa_2 e^{-\gamma^c (T-t)}}, \\ A^c(t,T) &= \frac{2\theta_r}{\sigma_r^2} \log \left[\frac{\gamma^c e^{\kappa_2 \cdot (T-t)}}{\kappa_1 - \kappa_2 \cdot e^{-\gamma^c \cdot (T-t)}} \right] \end{aligned}$$

with $\gamma^c := \sqrt{\hat{a}_r^2 + 2\sigma_r^2 c}$, $\kappa_1 := \frac{\hat{a}_r}{2} + \frac{\gamma^c}{2}$ and $\kappa_2 := \frac{\hat{a}_r}{2} - \frac{\gamma^c}{2}$ solve the Riccati equations for $c \geq -\frac{\hat{a}_r^2}{2\sigma_r^2}$.

B Proof of Theorem 2.2:

After factoring out $C(t_k)$ in (9) we apply the approximation

$$e^{x+y+z} \approx 1 + x + y + z \approx e^x + y + z.$$

For our purposes, this is a sufficiently good approximation since the quantities corresponding to x, y, z in (9) are small. If we approximate the integrals by sums, we obtain:

$$\begin{aligned} E_{\widetilde{Q}} \left[e^{-\int_0^{t_k} (r(s) + p_{\text{refi}}(s)) ds} \right] &\approx C(t_k) \cdot \widetilde{P}(0, t_k) \\ -\beta b_{10} \cdot C(t_k) \cdot \Delta t \cdot \sum_{k=1}^{\lceil \frac{t_k}{\Delta t} \rceil} & E_{\widetilde{Q}} \left[\max \left(r(k \cdot \Delta t) - \frac{\text{WAC} + a_{10}}{b_{10}}, 0 \right) \right] \\ +\beta b_{10} \cdot C(t_k) \cdot \Delta t \cdot \sum_{k=1}^{\lceil \frac{t_k}{\Delta t} \rceil} & E_{\widetilde{Q}} \left[\max \left(\frac{\text{WAC} + a_{10} - \alpha}{b_{10}} - r(k \cdot \Delta t), 0 \right) \right] \\ &= C(t_k) \cdot \widetilde{P}(0, t_k) \\ -\beta b_{10} \cdot C(t_k) \cdot \Delta t \cdot \sum_{k=1}^{\lceil \frac{t_k}{\Delta t} \rceil} & \int_{r_{Cap}}^{\infty} (r(k \cdot \Delta t) - r_{Cap}) f(r(k \cdot \Delta t)) dr(k \cdot \Delta t) \\ +\beta b_{10} \cdot C(t_k) \cdot \Delta t \cdot \sum_{k=1}^{\lceil \frac{t_k}{\Delta t} \rceil} & \int_0^{r_{Floor}} (r_{Floor} - r(k \cdot \Delta t)) f(r(k \cdot \Delta t)) dr(k \cdot \Delta t), \end{aligned} \quad (17)$$

where $f(\cdot)$ denotes the pdf of the short rate. Since we work with a CIR model in this paper, we know from Cox et al. (1985) that the distribution of $2 \cdot c_k \cdot r(k \cdot \Delta t)$ is the non-central χ^2-distribution with parameters $2q+2$ and $2u_k$, with c_k, u_k and q as previously defined. From the recurrence relation (see Johnson et al. (1995), p. 442)

$$\begin{aligned} \lambda \cdot \chi^2(x;\mu+4,\lambda) &= (\lambda-\mu)\cdot\chi^2(x;\mu+2,\lambda) \\ &\quad +(x+\mu)\cdot\chi^2(x;\mu,\lambda) - x\cdot\chi^2(x;\mu-2,\lambda) \end{aligned} \tag{18}$$

(for $\mu > 2$) and from the relation (see Johnson et al. (1995), p. 443)

$$\frac{\partial \chi^2(x;\mu,\lambda)}{\partial x} = f(x;\mu,\lambda) = \frac{1}{2}(\chi^2(x;\mu-2,\lambda) - \chi^2(x;\mu,\lambda)) \tag{19}$$

it follows with some easy calculations that

$$\int_0^b x f(x;\mu,\lambda)dx = \mu\cdot\chi^2(b;\mu+2,\lambda) + \lambda\cdot\chi^2(b;\mu+4,\lambda). \tag{20}$$

Applying (20) to the first integral in (17), we obtain:

$$\begin{aligned} \int_{r_{Cap}}^{\infty} (r(k\cdot\Delta t) - r_{Cap})\, f(r(k\cdot\Delta t))dr(k\cdot\Delta t) &= \frac{1}{2c_k}\Bigg[E_{\widetilde{Q}}[2c_k r(k\cdot\Delta t)] - \\ & -(2q+2)\cdot\chi^2(2c_k r_{Cap}; 2q+4, 2u_k) - 2u_k\cdot\chi^2(2c_k r_{Cap}; 2q+6, 2u_k)\Bigg] \\ & -r_{Cap}\cdot(1-\chi^2(2c_k r_{Cap}; 2q+2, 2u_k)). \end{aligned}$$

Similarly,

$$\begin{aligned} \int_0^{r_{Floor}} (r_{Floor} - r(k\cdot\Delta t))\, f(r(k\cdot\Delta t))dr(k\cdot\Delta t) &= \\ r_{Floor}\cdot\chi^2(2c_k r_{Floor}; 2q+2, 2u_k) - \frac{1}{2c_k}\Bigg[(2q+2) \\ \cdot\chi^2(2c_k r_{Floor}; 2q+4, 2u_k) + 2u_k\cdot\chi^2(2c_k r_{Floor}; 2q+6, 2u_k)\Bigg]. \end{aligned}$$

Noting that

$$E_{\widetilde{Q}}[2c_k r(k\cdot\Delta t)] = 2q+2+2u_k,$$

formula (10) follows directly after rearranging of terms.

C Proof of Theorem 2.3:

From the Feynman-Kac representation of the Cauchy-Problem (see, e.g. Zagst (2002), [2.6]) we know that $P^{base}(t,T)$ must satisfy:

$$\begin{aligned} P_t^{base} + (\theta_w - \hat{a}_w w)P_w^{base} + (\theta_p + b_{pw}\cdot w - \hat{a}_p p_0)P_{p_0}^{base} \\ +\frac{1}{2}\cdot(\sigma_p^2\cdot P_{p_0p_0}^{base} + \sigma_w^2\cdot P_{ww}^{base}) = p_0\cdot P^{base} \end{aligned}$$

Calculating the derivatives of P^{base} it follows that

$$\begin{aligned}A_t^d(t,T) - p_0(1 - \hat{a}_p C^d(t,T) + C_t^d(t,T))\\ -w(D_t^d(t,T) - \hat{a}_w D^d(t,T) + b_{pw} C^d(t,T))\\ +\tfrac{1}{2}\cdot(\sigma_p^2 C^d(t,T)^2 + \sigma_w^2 D^d(t,T)^2) - \theta_p C^d(t,T) - \theta_w D^d(t,T) = 0.\end{aligned}$$

Thus, we obtain the system of linear differential equations

$$\begin{aligned} 1 - \hat{a}_p C^d(t,T) + C_t^d(t,T) &= 0\\ b_{pw} C^d(t,T) - \hat{a}_w D^d(t,T) + D_t^d(t,T) &= 0\\ A_t^d(t,T) + \frac{1}{2}\cdot(\sigma_p^2 C^d(s,T)^2 + \sigma_w^2 D^d(s,T)^2) &\\ -\theta_p C^d(s,T) - \theta_w D^d(s,T) &= 0\end{aligned}$$

with $A^d(T,T) = 0$, $C^d(T,T) = D^d(T,T) = 0$. With some easy calculations it is straightforward to verify that the formulas as stated in Theorem 2.3 are the solutions of the linear differential equations above.

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