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Stochastic Dominance of Portfolio Insurance Strategies

OBPI versus CPPI

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Abstract The purpose of this article is to analyze and compare two standard portfolio insurance methods: Option-based Portfolio Insurance (OBPI) and Constant Proportion Portfolio Insurance (CPPI). Various stochastic dominance criteria up to third order are considered. We derive parameter conditions implying the second- and third-order stochastic dominance of the CPPI strategy. In particular, restrictions on the CPPI multiplier resulting from the spread between the implied volatility and the empirical volatility are analyzed.

Key words Portfolio insurance, CPPI, OBPI, stochastic dominance, volatility spread, risk-averse investor

1 Introduction

In the last years, private retirement arrangements have become an issue of more and more importance to lots of investors. With this respect, customers usually demand a guaranteed minimum performance on their invested capital from the offering banks and insurance companies. Suitable investment strategies to provide this required guarantee are so-called portfolio insurance strategies. They provide downside protection in falling markets while keeping the potential of profit in rising markets at the same time. The variety of portfolio insurance models is wide as any rule that takes less risk at lower wealth levels and more risk at higher wealth levels is basically a candidate. However, this paper focuses on the two most prominent examples, the Constant Proportion Portfolio Insurance (CPPI) strategy and the Option-based Portfolio Insurance (OBPI) strategy.

The CPPI strategy was introduced by Perold (1986) (see also Perold and Sharpe (1988)) for fixed income instruments and Black and Jones (1987) for equity instruments. It has been further analyzed in Black and Rouhani (1989) and Black and Perold (1992). Basically, it implements a simple strategy to allocate assets dynamically over time. The option-based portfolio insurance strategies date from 1976, when H. Leland and M. Rubinstein were the firsts to think about put options for portfolio hedging reasons.[1] Basically, it consists of buying simultaneously a portfolio invested in a risky asset and a put option written on it. Whereas the CPPI strategy being a dynamic investment strategy requires continuous reallocation of the corresponding portfolio, the OBPI strategy represents a static investment strategy and thus no further rebalancing of the portfolio is necessary after the initial purchase of the protecting put option. It is therefore frequently discussed whether the comfort of the static OBPI comes at a price compared to the dynamically rebalancing CPPI if we want to guarantee a minimum performance over a given time horizon T.

Analyses of the two portfolio insurance strategies were already conducted in Black and Rouhani (1989), Black and Perold (1992) (for the CPPI method) and Bookstaber and Langsam (2000). Bertrand and Prigent (2005) compare the two methods with respect to various criteria, introducing systematically the probability distributions of the two portfolio values. They conclude that neither of the two strategies dominates the other one statewisely or stochastically in first order. The present paper extends their analysis in two different aspects. Similar to the previous analyses, we assume a standard Black-Scholes model for the underlying assets. However, we should not miss the fact that the two investment strategies act in different market environments. Whereas the CPPI strategy represents a dynamic investment strategy that operates on the financial market with its empirical market volatility, the OBPI uses put options with different exercise prices that have to be priced using the implied volatility. It is a well-known fact in the financial market that one usually observes a spread between the empirical and the implied volatility. As an example Figure (1) visualizes the intra-month volatility estimated from daily DJ Euro Stoxx 50 index returns[2] and the corresponding implied volatilities given by the VStoxx index.

[1] See Leland and Rubinstein (1988). Actually, Leland and Rubinstein didn't use put options in order to provide portfolio insurance, as these didn't exist at that time for entire portfolios. Instead, they replicated the put option according to the Black-Scholes formula and no-arbitrage arguments. This investment strategy is now known as the Synthetic Put Portfolio Insurance (SPPI) strategy.

[2] To estimate the intra-month volatility of the DJ Euro Stoxx 50 index, for each month we calculated the (annualized) standard deviation of the corresponding daily intra-month returns.

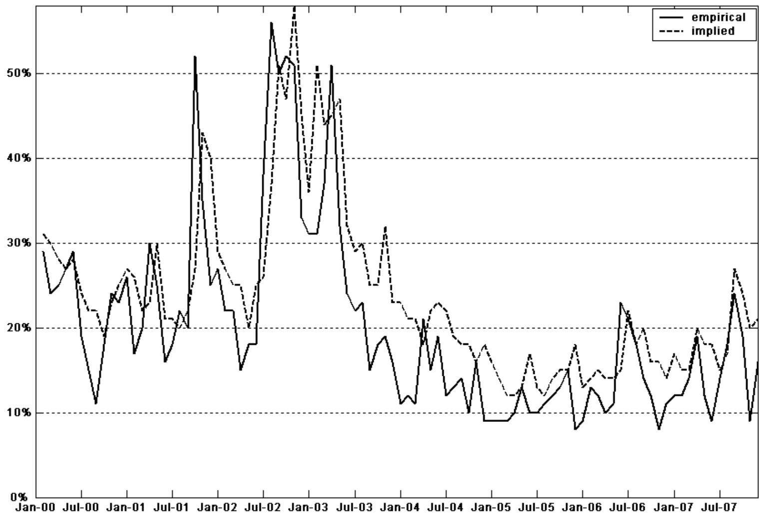

Fig. 1 Empirical and implied volatility. The empirical volatility is estimated from daily DJ Euro Stoxx 50 returns within each month. The implied volatility is given by the VStoxx index. The time period considered in the calculation is 01/2000-11/2007.

Since the implied volatility usually exceeds the empirical volatility, we actually have to pay a higher price for the put option used in the OBPI strategy compared to the Black-Scholes price based on the empirical volatility or the corresponding hedging strategy in the underlying market. Hence, the impact of the volatility spread should be considered in the performance analysis of the two strategies and will be one of the main focuses in our analysis.

Secondly, previous analyses only examined first-order stochastic dominance criteria, which is related to increasing utility functions. This signifies that the corresponding investors prefer more return to less return, whereas the associated risk is not taken into account. In general, however, we observe a certain saturation of the investor. Usually, the gain in utility from an additional unit decreases with the income level and these so-called risk-averse investors are described by increasing, concave utility functions. This is the reason why we extend the analysis of Bertrand and Prigent (2005) to stochastic dominance criteria up to third order. More precisely, we seek to deduce parameter conditions under which the CPPI strategy stochastically dominates the OBPI strategy. In all of our considerations we comprise the effect of the spread between the empirical and the implied volatility.

The remainder of this paper is organized as follows: In Section 2, we briefly introduce and discuss the two portfolio insurance strategies under consideration. We examine their final payoffs and compute their expectations and variances. Section 3 provides a theoretical comparison of the payoffs with respect to various criteria of stochastic dominance. The focus lies on second- and third-order stochastic dominance. To conclude the analysis Section 4 summarizes the main findings and gives some concluding remarks.

2 Basic properties of the CPPI and the OBPI strategy

2.1 The financial market

In order to compare the performances of the two portfolio insurance strategies, we start with defining the two strategies mathematically. We consider a classic Black-Scholes model where two basic assets are traded continuously in time during the investment period $[0,T]$. Within the context of the two portfolio insurance strategies the time horizon T can, e.g., be regarded as the time horizon for the given guarantee or the time of retirement. The first of the two assets is a risk-free asset, like a zero-coupon bond or cash-account, and is denoted by B. Its value grows with constant continuous interest rate $r > 0$ according to

$$B_t = B_0 \cdot e^{r \cdot t}, \tag{1}$$

and positive initial value $B_0 > 0$. The second asset, denoted by S, is subject to systematic risk, such as a stock, stock portfolio or market index. Now and in the following, we call S the risky asset and the stochastic dynamics of its market value are given by the geometric Brownian motion

$$dS_t = S_t \cdot (\mu dt + \sigma dW_t), \tag{2}$$

and positive initial value $S_0 > 0$. $W = (W_t)_{0 \le t \le T}$ is a standard Brownian motion and $\mu > r > 0$ and $\sigma > 0$ are constants that represent the drift[3] and the volatility of the asset price S, respectively. Then, following from Itô's lemma, the log-returns of the risky asset are normally distributed according to

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right) \cdot t, \sigma^2 \cdot t\right). \tag{3}$$

Within the scope of this paper we limit our considerations to self-financing investment strategies, i.e. strategies where money is neither injected nor

[3] Note that the assumption $\mu > r$ can easily be understood by observing that a typical risk-averse investor is characterized by a monotonely increasing, concave utility function u. Hence if $\mu < r$ there would be no reason for a rational investor to invest in stocks, since at any time $t \in [0,T]$

$$E\left[u\left(S_0 \cdot e^{r \cdot t}\right)\right] = u\left(S_0 \cdot e^{r \cdot t}\right) \ge u\left(S_0 \cdot e^{\mu \cdot t}\right) \ge u\left(E\left[S_t\right]\right) \ge E\left[u\left(S_t\right)\right].$$

withdrawn during the trading period $(0, T)$. Furthermore, following Black and Scholes (1973), we assume "ideal conditions" in the market for stocks and options. Markets are therefore frictionless and do not provide any arbitrage opportunities. Moreover, there are no transaction costs, taxes or margin requirements. Borrowing and short-selling as well as divisibility of shares are allowed without restriction. As the borrowing and lending rates are both assumed to be equal to the risk-free rate of return r, default risk is excluded. As far as options are considered, we restrict ourselves to European options that can only be exercised on a predetermined date. Furthermore, the underlying stocks do not pay dividends during the life of the option.[4] Whenever the price of an option has to be determined, we take account of the spread between the empirical and the implied volatility by using the Black-Scholes model (1),(2) with the implied volatility σ^i instead of the underlying empirical volatility σ.

In the following, $V^{IS} = \left(V_t^{IS}\right)_{0 \leq t \leq T}$ denotes the portfolio value process associated with the investment strategy IS. By means of simplicity we sometimes omit the index IS if one can conclude from the context which strategy is referred to. We start with a brief review of the CPPI strategy.

2.2 Constant Proportion Portfolio Insurance (CPPI)

The basic idea of the CPPI approach consists of managing a dynamic portfolio, so that its terminal value V_T^{CPPI} at the end of the investment horizon T lies above an investor-defined level F_T, given as a percentage $\alpha_T \geq 0$ of the initial investment V_0^{CPPI}, i.e.

$$F_T = \alpha_T \cdot V_0^{CPPI}. \tag{4}$$

Note that in the absence of any arbitrage opportunities it is impossible to find an investment that returns more than the risk-free rate of return r with no risk, and thus the maximum guaranteed portfolio value at the end of the investment period T is limited by

$$\alpha_T \leq e^{r \cdot T}. \tag{5}$$

Let $(F_t)_{0 \leq t \leq T}$ denote the present value of the guarantee, the so-called floor. By discounting with the risk-free rate of return r, it evolves according to

$$F_t = \alpha_t \cdot V_0^{CPPI}, \quad \alpha_t = \alpha_T \cdot e^{-r \cdot (T-t)}. \tag{6}$$

The surplus of the current portfolio value V_t^{CPPI} over the floor F_t is called cushion C_t and its value at any time $t \in [0, T]$ is given by

$$C_t = \max\left\{V_t^{CPPI} - F_t, 0\right\}. \tag{7}$$

[4] Note that in the case of a stock, modeled as a geometric Brownian motion, that pays dividends continuously over time at a constant rate d per unit time, the drift term μ in Equation (2) is simply replaced by the drift $\mu - d$ and all the other calculations remain the same (see, e.g., Shreve (2004)).

In order to ensure a minimum final portfolio value $V_T^{CPPI} \geq F_T$, the basic idea of the CPPI method now consists of investing a constant proportion m of the cushion C_t in the risky asset. This is the reason why the strategy is called constant proportion portfolio insurance. The investment in the risky asset is called exposure $(E_t)_{0 \leq t \leq T}$ and is determined by

$$E_t = m \cdot C_t = m \cdot \max\left\{V_t^{CPPI} - F_t, 0\right\}. \tag{8}$$

The remaining part of the portfolio

$$B_t = V_t^{CPPI} - E_t$$

is invested in the riskless asset. Notice that the payoff function is convex if the so-called multiplier m satisfies $m \geq 1$. By applying Itô's lemma, the value of the CPPI portfolio V_t^{CPPI} at any time t during the investment horizon $[0, T]$ can be derived as[5]

$$\begin{aligned} V_t^{CPPI} &= F_t + C_t = \alpha_t \cdot V_0^{CPPI} + C_t \qquad (9) \\ &= \alpha_T \cdot e^{-(T-t)r} \cdot V_0^{CPPI} + C_0 \cdot \left(\frac{S_t}{S_0}\right)^m \cdot e^{(1-m)\left(r+\frac{1}{2}m\sigma^2\right)t}. \end{aligned}$$

Thus, the CPPI method is parametrized by the level of insurance α_T and the multiplier m. Note that the value of the CPPI portfolio V_t^{CPPI} is always above the current floor $F_t = \alpha_t \cdot V_0^{CPPI}$ as $C_t > 0$. Hence, the floor F_t represents the dynamically insured amount of the portfolio. Furthermore, from Equation (9) we can see that the value process of the CPPI strategy is path-independent, i.e. does not depend on the stock price evolution in the investment period $(0, t)$.[6]

We conclude the section with the determination of the expected value as well as the variance of the value of the CPPI portfolio V_T^{CPPI} at the end of the investment horizon T, which will be needed in the upcoming stochastic dominance analysis.

Proposition 1 *The mean and the variance of the CPPI portfolio value at the end of the investment period T are given by*

$$\mu\left(V_T^{CPPI}\right) = E\left[V_T^{CPPI}\right] = \alpha_T \cdot V_0^{CPPI} + C_0 \cdot e^{[r+m\cdot(\mu-r)]\cdot T}, \tag{10}$$

$$\sigma^2\left(V_T^{CPPI}\right) = C_0^2 \cdot e^{2\cdot[r+m\cdot(\mu-r)]\cdot T} \cdot \left(e^{m^2\sigma^2 T} - 1\right), \tag{11}$$

where $C_0 = V_0^{CPPI} \cdot \left(1 - \alpha_T \cdot e^{-r\cdot T}\right)$.

Proof See the Appendix A.

[5] Details about this formula are provided in the Appendix (see also Bertrand and Prigent (2005)).

[6] This important property of the CPPI strategy was earlier shown by Bertrand and Prigent (2005).

Note that the expected terminal value of the CPPI strategy is independent on the stock price volatility σ. In contrast, its volatility grows exponentially with the market volatility σ, which can be intensified by a high value of the multiplier m. An increase in the desired level of insurance α_T obviously reduces the investment risk $\sigma^2\left(V_T^{CPPI}\right)$. However, the expected portfolio value $\mu\left(V_T^{CPPI}\right)$ is decreased at the same time. Opposite effects can be observed with respect to the choice of the multiplier m, which determines the portfolio's participation in the stock market.

Next, we will give a short description of the protective put strategy as an example for an option-based portfolio insurance strategy.

2.3 Option-based Portfolio Insurance (OBPI)

In contrast to the CPPI strategy, the OBPI strategy is a static investment strategy. It basically guarantees a minimum terminal portfolio value of $V_T^{OBPI} = \alpha_T \cdot V_0^{OBPI}$ for a portfolio consisting of q shares of the risky asset S, by purchasing European put options with maturity T and strike price X on the same number of shares. To simplify our presentation, we assume that q is normalized and set equal to one and that the put option is leverage-financed at the risk-free interest rate r at inception $t = 0$. The corresponding loan will be refunded at maturity T. At inception $t = 0$, the total portfolio value is then given by

$$V_0^{OBPI} = S_0 + Put\left(S_0, X, r, \sigma^i, 0, T\right) - Put\left(S_0, X, r, \sigma^i, 0, T\right) = S_0,$$

where $Put\left(S_t, X, r, \sigma^i, t, T\right)$ denotes the Black-Scholes value of a European put option (value of the underlying asset S_t, strike price X, risk-free rate of return r, implied volatility σ^i, valuation time $t \leq T$, maturity T). Since the OBPI strategy is a static investment strategy, no trading takes place during the investment period $(0, T)$. Hence, the final portfolio value V_T^{OBPI} at maturity T is given by

$$\begin{aligned} V_T^{OBPI} &= S_T + Put\left(S_0, X, r, \sigma^i, T, T\right) - Put\left(S_0, X, r, \sigma^i, 0, T\right) \cdot e^{r \cdot T} \\ &= \max\left\{X, S_T\right\} - Put\left(S_0, X, r, \sigma^i, 0, T\right) \cdot e^{r \cdot T}. \end{aligned} \tag{12}$$

In order to guarantee a minimum terminal portfolio value of $V_T^{OBPI} = \alpha_T \cdot V_0^{OBPI}$, the strike X of the hedging European put option must equal[7]

$$X = Put\left(S_0, X, r, \sigma^i, 0, T\right) \cdot e^{r \cdot T} + \alpha_T \cdot V_0, \quad S_0 = V_0^{OBPI}. \tag{13}$$

Notice, that similar to the restriction of the insurance level α_T in the case of the CPPI strategy by (5), relation (13) also caps the maximum guaranteed portfolio value for an OBPI strategy. Generally, in contrast to the CPPI approach, the strike price X of the hedging put option (which depends

[7] The corresponding strike price X can be determined from this equation by a zero search method, e.g. the Newton method.

on the desired level of insurance α_T) represents the only parameter of the OBPI strategy. For simplicity, we presume in the following analysis that the required European put option $Put\left(S_0, X, r, \sigma^i, 0, T\right)$ is available in the (OTC) market.[8] To simplify the notation, now and in the following, we often use the notation

$$\begin{aligned} Put\left(S_0, r, \sigma^i\right) &:= Put\left(S_0, X, r, \sigma^i, 0, T\right), \\ Call\left(S_0, r, \sigma^i\right) &:= Call\left(S_0, X, r, \sigma^i, 0, T\right), \end{aligned}$$

when the underlying strike price X, and the inception and terminal date, 0 and T, respectively, are clear from the context. With respect to the hedging put option $Put\left(S_0, X, r, \sigma^i, 0, T\right)$ we also use the abbreviation

$$Put_0 := Put\left(S_0, X, r, \sigma^i, 0, T\right).$$

Similar to the CPPI strategy, we finally determine the expected value as well as the variance of the terminal value of the OBPI portfolio V_T^{OBPI} at maturity T. For this purpose, we recall the definition of lower and upper partial moments.

Definition 1 *Given the benchmark X and a random variable S, the Lower Partial Moment LPM_z and the Upper Partial Moment UPM_z of S with respect to X and $z \in \mathbb{N}_0$ is defined as*

$$LPM_z(S, X) = E\left[\max\{X - S, 0\}^z\right], \tag{14}$$

$$UPM_z(S, X) = E\left[\max\{S - X, 0\}^z\right]. \tag{15}$$

In terms of an asset price S and a corresponding benchmark X the lower partial moment LPM_0 represents the shortfall probability and LPM_1 the expected value of the loss, when the asset price falls below the benchmark. Vice versa, UPM_0 denotes the probability of outperformance and UPM_1 the expected value of the profit in the case when the asset price beats the benchmark X. Based on these definitions, the mean and the variance of the terminal portfolio value of an OBPI strategy can be determined as follows.

Proposition 2 *The mean and the variance of the value of the OBPI portfolio at maturity T are given by*

$$\mu\left(V_T^{OBPI}\right) = E\left[V_T^{OBPI}\right] = UPM_1(S_T, X) + \alpha_T \cdot V_0^{OBPI}, \tag{16}$$

$$\sigma^2\left(V_T^{OBPI}\right) = UPM_2(S_T, X) - UPM_1(S_T, X)^2. \tag{17}$$

Proof See the Appendix B.

[8] Note that this is not a very restrictive assumption, since the investment horizon T is typically very long and the underlying OTC market offers European put options of virtually any maturity.

Notice that an increase in the desired level of insurance α_T, or correspondingly in the strike X, results in a lower call premium of the call option with payoff $\max\{S_T - X, 0\}$ that corresponds to the upper partial moment $UPM_1(S_T, X)$. This reduces the exercise probability of the call option and thus the value of the upper partial moment in Equation (16). Correspondingly, the expected terminal value $\mu\left(V_T^{OBPI}\right)$ of the OBPI strategy decreases with an increase in the level of insurance α_T and, at the same time, the variance of the terminal value $\sigma^2\left(V_T^{OBPI}\right)$ decreases.

Based on the deduced payoffs and distribution characteristics of the two investment strategies under consideration, we can now proceed with the comparison of the two strategies using stochastic dominance criteria.

3 CPPI versus OBPI

In order to compare the two methods, the initial portfolio values V_0^{CPPI} and V_0^{OBPI} are assumed to equal the current value of the risky asset S_0, i.e.

$$V_0 := V_0^{CPPI} = V_0^{OBPI} = S_0.$$

Also, the two strategies are supposed to provide the same guarantee α_T at the end of the (finite) investment period[9] T expressed as proportion of the initial investment V_0.[10] Hence,

$$\alpha_t = \alpha_T \cdot e^{-r\cdot(T-t)}, \quad F_t = \alpha_t \cdot V_0,$$

in the case of the CPPI strategy and the strike price of the European put option for the OBPI strategy satisfies

$$X = Put_0 \cdot e^{r\cdot T} + \alpha_T \cdot V_0.$$

Note that these two conditions do not impose any constraint on the multiplier m. In what follows, this leads us to consider CPPI strategies for various values of the multiplier m.[11]

[9] Note that if $T \to \infty$ the floor of the CPPI strategy converges to zero

$$\alpha_t = \alpha_T \cdot e^{-r\cdot(T-t)} \to 0,$$
$$F_t = 0,$$

and thus results in a constant mix strategy with leverage factor $m-1$

$$E_t = m \cdot C_t = m \cdot V_t,$$
$$B_t = (1-m) \cdot V_t.$$

Hence, no minimum portfolio value is guaranteed anymore and no puts are needed to insure the portfolio.

[10] It is our understanding, that the insurance level α_T satisfies the Constraints (5) and (13).

[11] The multiple, however, must not be too high as shown for example in Bertrand and Prigent (2002 or 2005).

Let us start with looking at the payoff functions of both strategies. In the simplest case, one of the payoff functions of the two strategies would statewisely dominate the other one, which would imply that one of the strategies results for all S_T values in a higher terminal value than the other one. However, Bertrand and Prigent (2005) argue that, since $V_0^{CPPI} = V_0^{OBPI}$ and due to the absence of arbitrage, this is not possible, which leads to the following proposition.

Proposition 3 *Neither of the two payoffs is greater than the other for all terminal values S_T of the risky asset. The two payoff functions intersect one another.*

This finding can be illustrated using a simple numerical example with typical values for the financial market: $\mu = 7.50\%$, $\sigma = 15\%$, $\sigma^i = 18\%$ and $r = 3.5\%$. In this market, the two portfolio insurance strategies are set up assuming $T = 5$ (years), $V_0 = S_0 = 100$ and $\alpha_T = 103.5\%$. If not mentioned otherwise, now and in the following, we consider this setting as our reference model scenario for numerical calculations. The value of the CPPI strategy is calculated for different values of the multiplier $m = 1, 2, 3, 4, 5$. Figure (2) visualizes the obtained terminal values of the two strategies dependent on the terminal value S_T of the risky asset. Notice that a more theoretical motivation of Proposition 3 will be given in the proof of Theorem 3.

For each value of the multiplier m the payoffs of the CPPI and the OBPI strategy intersect at least once.

Since we could not observe a simple dominance of one of the two strategies, we will consider more sophisticated criteria of stochastic dominance starting from first- up to third-order in the sequel.

3.1 First-order stochastic dominance

In general, stochastic dominance criteria try to rank two random variables V and V^* according to special classes of utility functions $\mathbb{U}$.[12] It is said, that the random variable V^* stochastically dominates the random variable V with respect to $\mathbb{U}$, i.e. $V \prec_{\mathbb{U}} V^*$, if and only if

$$E\left[u\left(V\right)\right] \leq E\left[u\left(V^*\right)\right],$$

for all $u \in \mathbb{U}$ for which the two expected values exist. In the case of first-order stochastic dominance $\mathbb{U}$ is the class of all real, measurable and increasing functions denoted by

$$\mathbb{U}_1 := \{u : \mathbb{R} \to \mathbb{R} : \ u \text{ measurable}, \ u' \geq 0\}.$$

[12] For further details concerning the concept of stochastic dominance, see, e.g., Mosler (1982).

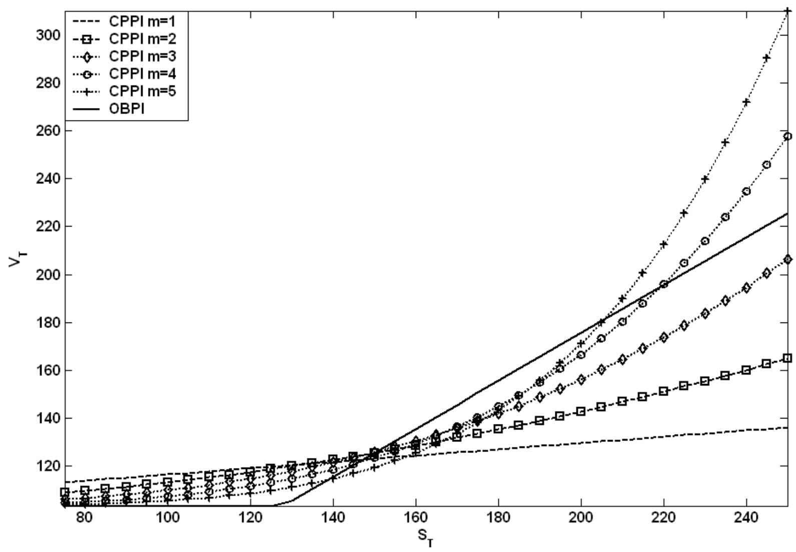

Fig. 2 CPPI and OBPI payoffs as functions of S_T, where $m = 1, 2, 3, 4, 5$ and $T = 5$ $(years)$, $\alpha_T = 103.5\%$, $V_0 = S_0 = 100$, $\mu = 7.50\%$, $\sigma = 15\%$, $\sigma^i = 18\%$ and $r = 3.50\%$.

This can be interpreted that investors like more money rather than less money and are non-satiated. Recall that a common criterion to test for the first-order stochastic dominance of the random variable V^* is to compare the cumulative distribution functions F_V and F_{V^*} of the two random variables. The random variable V^* stochastically dominates the random variable V in first order ($V \prec_{\mathbb{U}_1} V^*$ or briefly $V \prec_1 V^*$), if and only if for any outcome x the random variable V^* gives a higher probability of receiving an outcome equal to or better than x compared to V. Hence,

$$V \prec_1 V^* \Leftrightarrow F_{V^*}(x) \leq F_V(x), \quad \forall x \in \mathbb{R}. \tag{18}$$

With respect to the CPPI and the OBPI strategy Bertrand and Prigent (2005) show that neither of the two strategies stochastically dominates the other one at first order. However, first-order stochastic dominance represents the strongest criterion, i.e. it implies second- and third-order stochastic dominance. We therefore extend their analysis to the weaker principles of second- and third-order stochastic dominance. In particular, we try to find special conditions for which the CPPI strategy stochastically dominates the OBPI strategy. With this respect, the multiplier m, that determines the risk exposure, will be our most important parameter.

3.2 Second-order stochastic dominance

In comparison to first-order stochastic dominance, the second-order stochastic dominance criterion restricts to risk-averse investors. As mentioned earlier, investors described by utility functions $u \in \mathbb{U}_1$ are non-satiated, which means that their utility is strictly monotone increasing in the income level without taking account of the associated risk. However, in the financial market we traditionally observe a different behavior of the investor. Again, more money is preferred to less money. Nevertheless, the gain in utility from an additional unit decreases with the income level. This behavior is represented by the class of increasing, concave utility functions, denoted by

$$\mathbb{U}_2 := \{u : \mathbb{R} \longrightarrow \mathbb{R} : u \in \mathbb{U}_1 \text{ and } u'' \leq 0\},$$

and we say that the random variable V^* stochastically dominates the random variable V in second order if $V \prec_{\mathbb{U}_2} V^*$ or briefly $V \prec_2 V^*$.

Our goal in this section is to deduce conditions for the parameters of the two portfolio insurance strategies such that the CPPI strategy stochastically dominates the OBPI strategy in second order at the due-date T for the given guarantee. For this purpose, Mosler (1982) provides a useful criteria using intersection conditions, that is independent from any specific utility function $u \in \mathbb{U}_2$.

Theorem 1 (Mosler (1982)) *Let V, V^* be two random variables with finite expectation. Furthermore, let $H(x) := F_V(x) - F_{V^*}(x)$ for all $x \in \mathbb{R}$. Then,*

$$H \in \mathbb{S}_1, \quad E[V] \leq E[V^*] \quad \Rightarrow \quad V \prec_2 V^*.$$

Proof See Mosler (1982).

$\mathbb{S}_k$ describes the set of all real functions H with k changes of sign, i.e.

$$\mathbb{S}_k := \left\{ \begin{array}{l} H : \mathbb{R} \to \mathbb{R} : \exists s_1, ..., s_k \in \mathbb{R},\, s_0 := -\infty,\, s_{k+1} := +\infty, \\ \text{where } \ (-1)^j \cdot H(s) \geq 0, \forall s \in (s_j, s_{j+1}), j = 0, ..., k, \quad H \neq 0 \end{array} \right\}.$$

Example 1

$$\mathbb{S}_1 = \left\{ \begin{array}{l} H : \mathbb{R} \to \mathbb{R} : \exists s_1 \in \mathbb{R}, \\ \text{where } H(s) \left\{ \begin{array}{ll} \geq 0, & s \in (-\infty, s_1) \\ \leq 0, & s \in (s_1, \infty) \end{array} \right. , \quad H \neq 0 \end{array} \right\},$$

$$\mathbb{S}_2 = \left\{ \begin{array}{l} H : \mathbb{R} \to \mathbb{R} : \exists s_1, s_2 \in \mathbb{R}, \\ \text{where } H(s) \left\{ \begin{array}{ll} \geq 0, & s \in (-\infty, s_1) \\ \leq 0, & s \in (s_1, s_2) \\ \geq 0, & s \in (s_2, \infty) \end{array} \right. , \quad H \neq 0 \end{array} \right\}.$$

In terms of the cumulative distribution functions $F_V(x)$ and $F_{V^*}(x)$, the condition $H(x) := F_V(x) - F_{V^*}(x) \in \mathbb{S}_k$ implies that the two functions intersect exactly k-times.

In order to derive conditions for the second-order stochastic dominance of the CPPI strategy, we analyze the two conditions postulated in Theorem 1.

Theorem 2 *The following statements are equivalent:*

1. $E\left[V_T^{OBPI}\right] \leq E\left[V_T^{CPPI}\right]$.
2. $Call\left(S_0, r, \sigma^i\right) \cdot e^{(m-1)\cdot(\mu-r)\cdot T} \geq Call\left(S_0, \mu, \sigma\right)$, *i.e.*

$$m \geq 1 + \frac{1}{(\mu - r)\cdot T} \cdot \ln\left(\frac{Call\left(S_0, \mu, \sigma\right)}{Call\left(S_0, r, \sigma^i\right)}\right) =: m^1_{\min}. \tag{19}$$

Proof See the Appendix C.1.

Theorem 3 *Let* $m > 1$ *and* $H(x) := F_{V_T^{OBPI}}(x) - F_{V_T^{CPPI}}(x), \ \forall x \in \mathbb{R}$. *Then,*

$$(S_2): \frac{1}{m-1} \cdot \left(\frac{\left(1 - \alpha_T \cdot e^{-r\cdot T}\right)\cdot m}{e^{\frac{1}{2}\cdot(m-1)\cdot\sigma^2\cdot T}}\right)^{\frac{m}{m-1}} < \frac{C_0}{X \cdot e^{-r\cdot T}} \Rightarrow H \in \mathbb{S}_2.$$

If $m = 1$, $H \in \mathbb{S}_1$ *is true.*

Proof See the Appendix C.2.

Remark 1 Note that using standard algebraic calculus one can easily show, that for m large the left hand side of Condition (S_2) is smaller than the right hand side and thus Condition (S_2) is satisfied.

Based on the relationship $H \in \mathbb{S}_1$, if $m = 1$, following from Theorem 3 and the constraint on the call prices resulting from Theorem 2 to provide $E\left[V_T^{CPPI}\right] \geq E\left[V_T^{OBPI}\right]$, we can directly conclude from Theorem 1 the second-order stochastic dominance of the CPPI strategy for $m = 1$.

Theorem 4 *Let* $m = 1$ *and* $Call\left(S_0, r, \sigma^i\right) \geq Call\left(S_0, \mu, \sigma\right)$. *Then,*

$$V_T^{OBPI} \prec_2 V_T^{CPPI}.$$

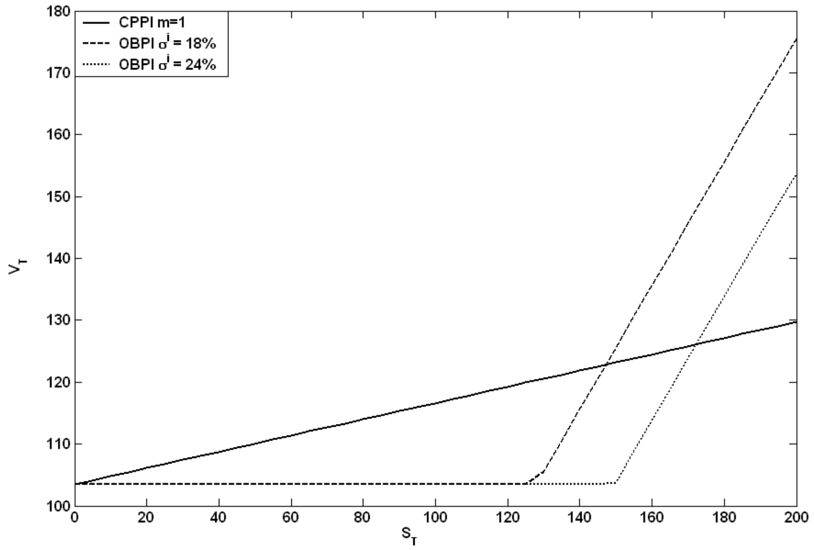

Fig. 3 Payoff functions of an OBPI and a CPPI strategy for different values of the implied volatility $\sigma^i = 18\%, \sigma^i = 24\%$ and $T = 5 \quad (years)$, $\alpha_T = 103.5\%, m = 1$, $V_0 = S_0 = 100$, $\mu = 7.50\%$, $\sigma = 15\%, r = 3.50\%$.

Table 1 Call prices based on $T = 5(years), \alpha_T = 103.5\%, m = 1, V_0 = S_0 = 100, \mu = 7.50\%, \sigma = 15\%, r = 3.50\%$.

$\mathbf{m = 1}$	**X**	**Call** (S_0, r, σ^i)		**Call** (S_0, μ, σ)
σ^i= **18**%	127.87	13.12	$<$	19.48
σ^i= **24**%	149.65	13.12	$>$	12.13

Remark 2 From Theorem 4 we conclude that in times of low expected return forecasts and high implied volatility in comparison to the empirical volatility the CPPI strategy will stochastically dominate the OBPI strategy in second order.

Figure (3) visualizes the statement of Theorem 4. The graph illustrates the payoff functions of the OBPI as well as the standard CPPI strategy for different values of the implied volatility σ^i. Table (1) provides the corresponding call prices according to Theorem 2 and 4, respectively.

From Table (1) and Theorem 4 we conclude that for $m = 1$ and $\sigma^i = 24\%$ the CPPI strategy stochastically dominates the OBPI investment strategy in second order. If $\sigma^i = 18\%$, the strike of the hedging put option used in the realization of the OBPI strategy is smaller. Consequently, the put option is not as expensive as in the case of a higher implied volatility of $\sigma^i = 24\%$. Theorem 2 tells us, that in this case the expected OBPI return exceeds that of the CPPI strategy (i.e. $E\left[V_T^{CPPI}\right] \leq E\left[V_T^{OBPI}\right]$).

Remark 3 $Call\left(S_0, X, r, \sigma^i, 0, T\right)$ is independent of the choice of the parameter σ^i. From put-call-parity follows

$$Call\left(S_0, X, r, \sigma^i, 0, T\right) = Put\left(S_0, X, r, \sigma^i, 0, T\right) + S_0 - X \cdot e^{-r \cdot T}.$$

Furthermore, since $X = \alpha_T \cdot V_0 + Put_0 \cdot e^{r \cdot T}$ we obtain

$$Call\left(S_0, X, r, \sigma^i, 0, T\right) = S_0 - \alpha_T \cdot V_0 \cdot e^{-r \cdot T}.$$

Theorem 4 provides a stochastic dominance criterion in the special case $m = 1$. In order to derive an analogue criterion for the more general case $m > 1$ we analyze third-order stochastic dominance. As already mentioned, third-order stochastic dominance follows from second-order stochastic dominance and further cuts down the class of utility functions $\mathbb{U}_3$ under consideration.

3.3 Third-order stochastic dominance

Third-order stochastic dominance adds ruin aversion to the risk aversion involved in second-order stochastic dominance. Investors prefer positive to negative skewness. Notice that portfolio insurance strategies, like the CPPI or the OBPI strategy, are characterized by providing downside protection while still participating in upside markets. Mathematically, the additional ruin aversion is expressed by requiring $u''' \geq 0$. Hence, the corresponding class of utility functions $\mathbb{U}_3$ is given by

$$\mathbb{U}_3 := \{u : \mathbb{R} \longrightarrow \mathbb{R} : u \in \mathbb{U}_2 \text{ and } u''' \geq 0\},$$

and we say that the random variable V^* stochastically dominates the random variable V in third order if $V \prec_{\mathbb{U}_3} V^*$ or briefly $V \prec_3 V^*$.

In particular, $\mathbb{U}_3$ includes the class of utility functions $\mathbb{U}_{\mathbb{DARA}}$ and $\mathbb{U}_{\mathbb{HARA}}$ providing $\mathbb{D}$ecreasing $\mathbb{A}$bsolute $\mathbb{R}$isk $\mathbb{A}$version ($\mathbb{DARA}$) and $\mathbb{H}$yperbolic $\mathbb{A}$bsolute $\mathbb{R}$isk $\mathbb{A}$version ($\mathbb{HARA}$), respectively. Here, absolute risk aversion is measured by the Arrow-Pratt measure of absolute risk aversion

$$ARA\,(v) := -\frac{u''(v)}{u'(v)},$$

and describes the investor's willingness to cover, based on her current wealth v, risks by paying an insurance premium. Then, the subsets $\mathbb{U}_{\mathbb{DARA}}, \mathbb{U}_{\mathbb{HARA}} \subseteq \mathbb{U}_3$ are defined as

$$\mathbb{U}_{\mathbb{DARA}} := \left\{ \begin{array}{l} u \in \mathbb{U}_3 : u'(v) = u'(a) \cdot e^{-\int_a^v r(z)dz}, u'(a) \geq 0, \\ a \in \mathbb{R},\ r \geq 0, r' \leq 0 \end{array} \right\},$$
$$\text{i.e. } ARA\,(v) = r\,(v)\,, \tag{20}$$

$$\mathbb{U}_{\mathbb{HARA}} := \left\{ \begin{array}{l} u \in \mathbb{U}_3 : u(v) = a + b \cdot \frac{(v-c)^\gamma}{\gamma}, v \geq c,\ \gamma < 1,\ b > 0, \\ a,\ c \in \mathbb{R} \end{array} \right\},$$

$$\text{i.e. } ARA(v) = \frac{1-\gamma}{v-c}. \tag{21}$$

Here, the absolute risk aversion is a decreasing function in wealth v. Hence, the higher her wealth v the less willing is the investor to hedge higher risks.[13] According to Elton and Gruber (1995) common investors are usually described by the class of $\mathbb{HARA}$ utility functions.

Similar to the previously analyzed second-order stochastic dominance, our goal is to deduce conditions for the parameters of the two portfolio insurance strategies such that the CPPI strategy stochastically dominates the OBPI strategy in third order at the end of the investment period T. Again, Mosler (1982) and Karlin and Novikov (1963), respectively, provide useful criteria using an intersection condition $\mathbb{S}_2$, that is independent from any utility function $u \in \mathbb{U}_3$.

Theorem 5 (Karlin, Novikov (1963), Mosler(1982)) *Let V, V^* be non-negative with finite second moment. Furthermore, let $H(x) := F_V(x) - F_{V^*}(x)$ for all $x \in \mathbb{R}$. Then,*

$$H \in \mathbb{S}_2, \quad E[V] \leq E[V^*], \quad E\left[(V^*)^2\right] \leq E\left[V^2\right] \quad \Rightarrow \quad V \prec_3 V^*.$$

Proof See, e.g., Mosler (1982).

Sufficient parameter restrictions to assure the outperformance of the expected terminal value of the CPPI strategy (i.e. $E\left[V_T^{OBPI}\right] \leq E\left[V_T^{CPPI}\right]$) and $H \in \mathbb{S}_2$ are already provided by Theorem 2 and 3. Hence, in order to derive further parameter conditions implying the third-order stochastic dominance of the CPPI strategy according to the Karlin and Novikov Theorem (1963), we still have to analyze the condition $E\left[\left(V_T^{CPPI}\right)^2\right] \leq E\left[\left(V_T^{OBPI}\right)^2\right]$.

Theorem 6 *Let*

$$\begin{aligned} f_{\max}(m) &:= e^{\mu T} \cdot Call^2\left(S_0, r, \sigma^i\right) \cdot e^{2\cdot(m-1)\cdot(\mu-r)\cdot T + m^2\sigma^2 T} \\ &\quad + 2 \cdot \alpha_T \cdot S_0 \cdot Call\left(S_0, r, \sigma^i\right) \cdot e^{(m-1)(\mu-r)\cdot T}, \\ b &:= Call\left(S_0, \mu, \sigma\right) \cdot \left[S_0 \cdot \left(2 \cdot \alpha_T + e^{\mu T} \cdot (1+\Delta)\right) - X\right] \\ \Delta &:= \Delta\left(S_0, \mu, \sigma\right) := \frac{Call\left(S_0 e^{\sigma^2 T}, \mu, \sigma\right) - Call\left(S_0, \mu, \sigma\right)}{Call\left(S_0, \mu, \sigma\right)}, \end{aligned}$$

and

$$m_{\max} := f_{\max}^{-1}(b).$$

Then, the following statements are equivalent:

[13] See Arrow (1965).

Table 2 Analysis of the conditions for third-order stochastic dominance of the CPPI strategy resulting from Theorem 7 for different values of the multiplier m.

	$\mathbf{m = 1}$	$\mathbf{m = 2}$	$\mathbf{m = 3}$	$\mathbf{m = 4}$	$\mathbf{m = 5}$
$\mathbf{m}_{\min}$	2.98				
$\mathbf{m}_{\min} \leq \mathbf{m}$	×	×	✓	✓	✓
Condition (S_2)	–	✓	✓	✓	✓
$\mathbf{m}_{\max}$	3.05				
$\mathbf{m} \leq \mathbf{m}_{\max}$	–	✓	✓	×	×
3^{rd} **order dominance**	–	×	✓	×	×

1. $E\left[\left(V_T^{CPPI}\right)^2\right] \leq E\left[\left(V_T^{OBPI}\right)^2\right]$.
2. $m \leq m_{\max}$.

Proof See the Appendix D.

By combining Theorem 2, 3 and 6 we have now everything we need to conclude the third-order stochastic dominance of the CPPI strategy from Theorem 5.

Theorem 7 *Let* $m_{\min}^1$ *and* $m_{\max}$ *be defined as in Theorem 2 and Theorem 6 and*

$$m_{\min} := \max\left\{1, m_{\min}^1\right\}.$$

Furthermore, let Condition (S_2) *of Theorem 3 be satisfied. Then,*

$$m \in [m_{\min}, m_{\max}] \Rightarrow V_T^{OBPI} \prec_3 V_T^{CPPI}.$$

To get a better understanding of the statement of Theorem 7, we analyze the payoffs of the OBPI and the CPPI strategy for the parameterization visualized in Figure (2). More precisely, the underlying market parameters are those of the reference model[14] and the CPPI multiplier takes the values $m = 1, ..., 5$. Table (2) analyzes for each value of the multiplier the conditions for second-order (if $m = 1$) or third-order (if $m > 1$) stochastic dominance, resulting from Theorem 4 and Theorem 7, respectively.

Since $m_{\min} = 2.98$, we conclude from Theorem 4 that the CPPI strategy does not dominate the OBPI strategy in second order for $m = 1$. Furthermore, following from Theorem 7, we only observe third-order stochastic dominance in the case when $m = 3$.

To get a better understanding of what happens in the different parameterizations, Figure (4) visualizes the difference $V_T^{OBPI} - V_T^{CPPI}$ in the final payoffs of the two strategies for different terminal stock values S_T and different values of the multiplier m.

[14] $\mu = 7.50\%$, $\sigma = 15\%$, $\sigma^i = 18\%$, $r = 3.5\%$, $T = 5$ (years), $V_0 = S_0 = 100$, $\alpha_T = 103.5\%, X = 127.87$.

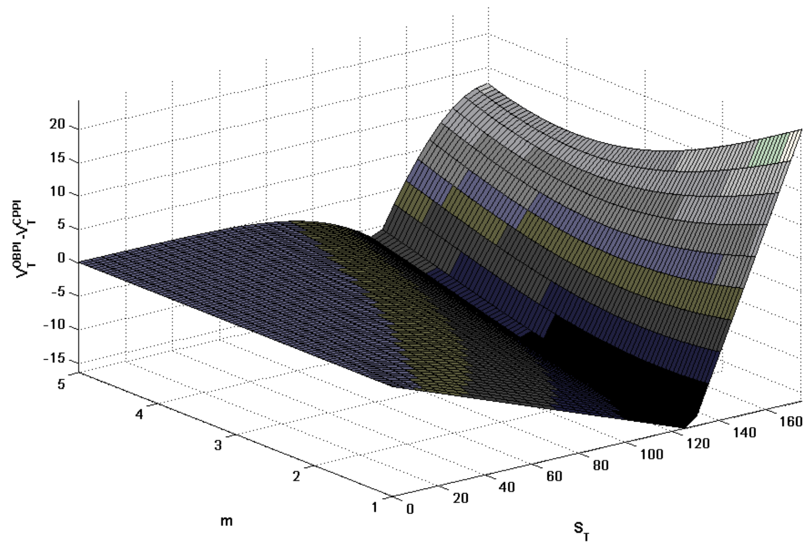

Fig. 4 Difference $V_T^{OBPI} - V_T^{CPPI}$ for the final payoffs of the two strategies, depending on the terminal stock price S_T as well as the multiplier m.

If $m = 1, 2$, the CPPI strategy is more likely to underperform the OBPI strategy, which results in $E\left[V_T^{OBPI}\right] > E\left[V_T^{CPPI}\right]$. In contrast, if $m = 3, 4, 5$ the expected terminal value of the CPPI strategy exceeds that of the OBPI strategy. However, the risk associated with the higher probability of outperformance exceeds that of the OBPI strategy, i.e. $E\left[\left(V_T^{CPPI}\right)^2\right] > E\left[\left(V_T^{OBPI}\right)^2\right]$ for $m = 4, 5$.

Finally, Figure (5), (6) and (7) more generally visualize the lower bound $m_{\min}$ and the upper bound $m_{\max}$ on the multiplier m resulting from Theorem 7 in dependence on the drift μ and the implied volatility σ^i of the underlying market, as well as the interval between the two bounds $m_{\max} - m_{\min}$. The remaining model parameters are given by the reference scenario.

From Figure (5) we conclude that the minimum multiplier $m_{\min}$ is the higher the lower the implied volatility σ^i. Since for low values of the implied volatility σ^i the hedging put option for the OBPI strategy is cheaper, the CPPI strategy must be allocated in a riskier fashion to outperform the protective put strategy. With respect to the drift μ, no definite dependence of the value of $m_{\min}$ can be observed.

Analogously, the upper bound $m_{\max}$ on the multiplier m decreases with an increase in the implied volatility σ^i (see Figure (6)). Since an increase in the implied volatility results in a higher premium for the put option used in the OBPI strategy at maturity T, the strike price X increases as well.

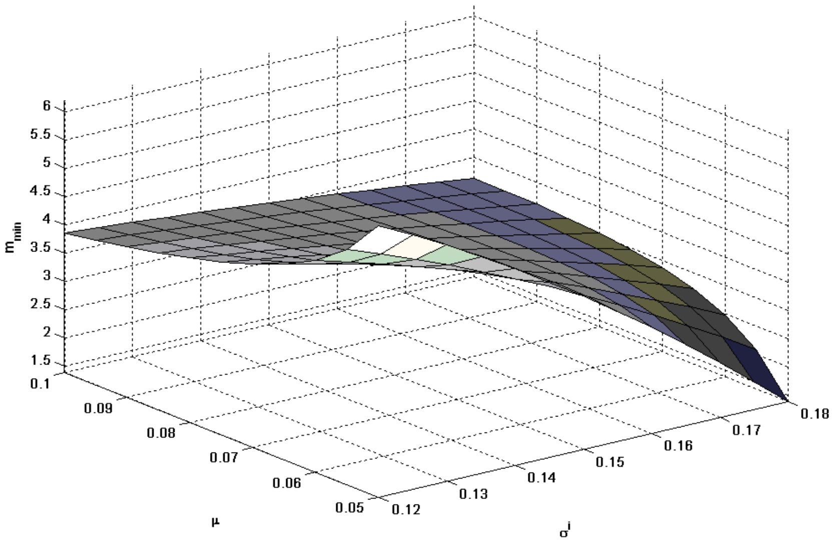

Fig. 5 Value of the threshold $m_{\min}$ as defined in Theorem 2 depending on the drift μ as well as the implied volatility σ^i of the underlying market.

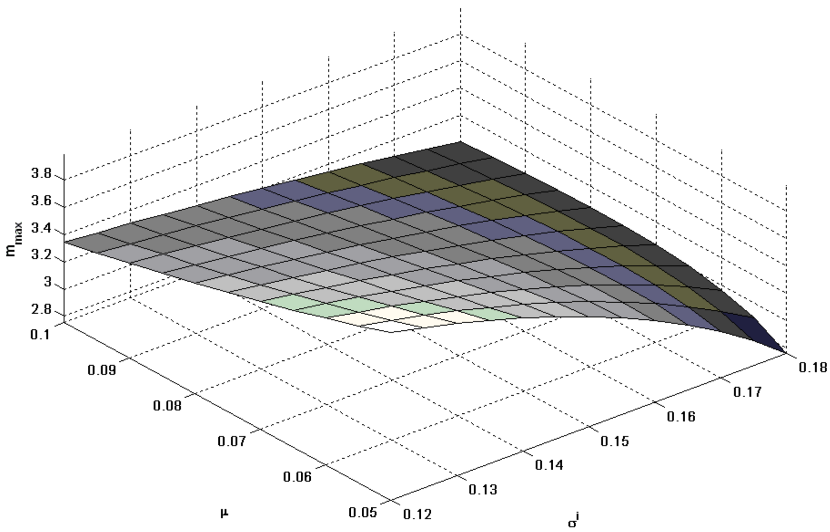

Fig. 6 Value of the threshold $m_{\max}$ as defined in Theorem 6 depending on the drift μ as well as the implied volatility σ^i of the underlying market.

Hence, at maturity T the put option is more likely to be exercised and thus the variance of the terminal value of the OBPI strategy $\sigma^2\left(V_T^{OBPI}\right)$ decreases with an increase in the implied volatility. In order for the risk associated with the CPPI strategy to be smaller than that of the OBPI strategy (which is exactly the interpretation of Statement 1 of Theorem 6), we now have to allocate the CPPI strategy in a more conservative fashion by using a smaller multiplier m.

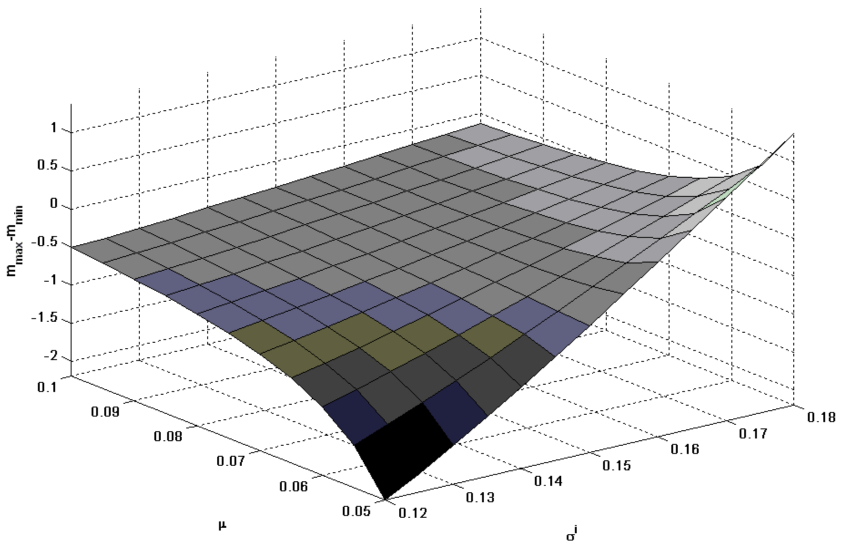

Fig. 7 Difference of the upper and the lower bound $m_{\max}-m_{\min}$ on the multiplier as defined in Theorem 7 depending on the drift μ as well as the implied volatility σ^i of the underlying market.

Finally, from Figure (7) we conclude, that there exist parameterizations of the financial market so that the interval $[m_{\min}, m_{\max}]$, derived in Theorem 7, actually includes admissible values for the multiplier m. Additionally, Figure (8) visualizes the difference

$$\frac{C_0}{X \cdot e^{-r \cdot T}} - \frac{1}{m-1} \cdot \left(\frac{\left(1 - \alpha_T \cdot e^{-r \cdot T}\right) \cdot m}{e^{\frac{1}{2} \cdot (m-1) \cdot \sigma^2 \cdot T}} \right)^{\frac{m}{m-1}}$$

corresponding to Condition (S_2) of Theorem 3 in dependence on the value of the multiplier m as well as the implied volatlity σ^i. The underlying empirical volatility is assumed to be $\sigma = 15\%$. Condition (S_2) is satisfied whenever we observe a positive value of the function. As we can see from the figure, for common parameterizations of the underlying financial market Condition

(S_2) is always satisfied. Altogether, we conclude that the CPPI strategy stochastically dominates the OBPI strategy in third-order in times of high implied volatilities (compared to the empirical volatility).

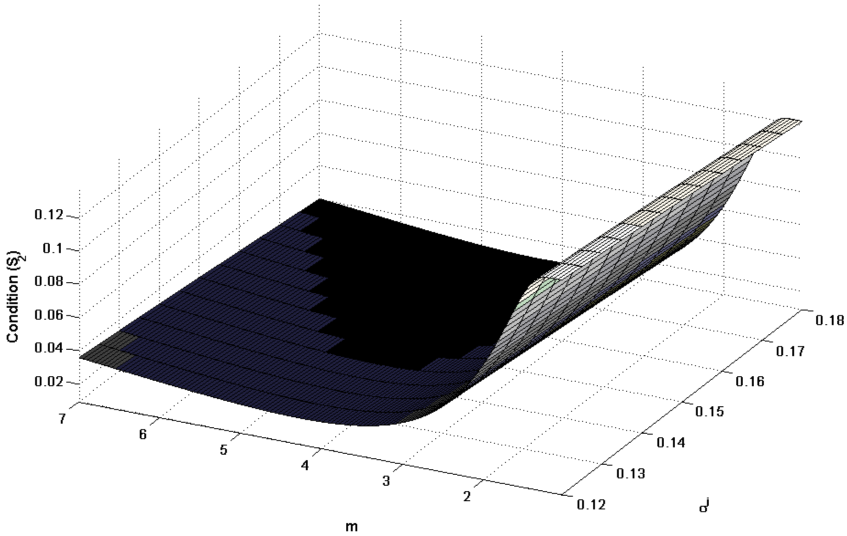

Fig. 8 Difference corresponding to Condition (S_2) of Theorem 3 in dependence on the value of the multiplier m as well as the implied volatility σ^i. The condition is satisfied, whenever we observe a positive value of the function.

To conclude our analysis of the CPPI and the OBPI strategy we will summarize the main results and give some concluding remarks.

4 Conclusion

In this paper, we have compared the two main portfolio insurance methods - the CPPI and the OBPI strategy - with respect to various criteria of stochastic dominance. With this respect, we have taken into account the impact of the spread between the (usually higher) implied volatility and the empirical volatility. Furthermore, we extended the work of previous papers by focussing our analysis on the relevant group of risk-averse investors that are described by increasing, concave utility functions. Although risk-averse investors prefer more money to less money, the gain in utility from an additional unit decreases with the income level.

In the past, neither statewise nor first-order stochastic dominance with respect to the terminal payoffs of the two strategies and increasing utility functions, respectively, could be confirmed. However, by considering risk-aversion in our stochastic dominance analysis we were able to derive specific conditions for the market parameters as well as the CPPI multiplier m implying the second- and third-order stochastic dominance of the CPPI strategy. More precisely, second-order stochastic dominance was based on the value $m = 1$, whereas we were able to derive an interval for the value of the multiplier m inducing third-order stochastic dominance. The resulting admissible multipliers significantly depend on the parameterization of the underlying financial market. More precisely, the CPPI strategy is more likely to stochastically dominate the OBPI strategy in third-order the higher the implied volatility σ^i.

So far we excluded the default risk of stocks and bonds in our analysis. The inclusion of default risk would result in a path-dependency of the CPPI strategy and will be subject of further research.

A Calculation of the CPPI value, mean and variance

With respect to the derivation of the value of the CPPI portfolio V_t we basically follow the proof of Bertrand and Prigent (2005). However, since for the derivation of the expected value and the variance we especially need the probability distribution of the cushion C_t, we briefly present the corresponding proof.

Recall that $V_t = C_t + F_t$, $E_t = mC_t$, $F_t = \alpha_t V_0$ and $d\alpha_t = \alpha_t r dt$. The value of the self-financing CPPI portfolio at time $t \in [0, T]$ is given by

$$\begin{aligned} dV_t &= (V_t - mC_t)\,\frac{dB_t}{B_t} + mC_t\frac{dS_t}{S_t} \\ &= [V_t - m\,(V_t - \alpha_t V_0)]\,\frac{B_t r dt}{B_t} + mC_t\frac{dS_t}{S_t} \\ &= [V_t\,(1-m) + m\alpha_t V_0]\,rdt + mC_t\frac{dS_t}{S_t}. \end{aligned}$$

Hence, the stochastic dynamics of the cushion C_t satisfy

$$\begin{aligned} dC_t &= d\,(V_t - \alpha_t V_0) = dV_t - V_0 d\alpha_t \\ &= [V_t\,(1-m) + m\alpha_t V_0]\,rdt + \frac{mC_t}{S_t}dS_t - V_0\alpha_t rdt \\ &= [(V_t - V_0\alpha_t)\,(1-m)]\,rdt + \frac{mC_t}{S_t}dS_t \\ &= C_t\,(1-m)\,rdt + \frac{mC_t}{S_t}dS_t. \end{aligned}$$

Substituting the geometric Brownian motion for the dynamics of the risky asset leads to

$$\frac{dC_t}{C_t} = [m\mu + r\,(1-m)]\,dt + m\sigma dW_t.$$

By applying Itô's lemma, it can be deduced that

$$\ln C_t - \ln C_0 = m\left(\ln S_t - \ln S_0\right) + (1-m)\left[r + \frac{1}{2}m\sigma^2\right] \cdot t. \qquad (22)$$

Thus,

$$C_t = C_0 \cdot \left(\frac{S_t}{S_0}\right)^m \cdot e^{(1-m)\left(r+\frac{1}{2}m\sigma^2\right)t}.$$

Substituting the lognormal distribution $\ln S_t \sim N(lnS_0 + (\mu - \frac{\sigma^2}{2})t, \sigma^2 t)$ for the risky asset S_t, we can deduce from (22) that the cushion C_t is lognormally distributed with

$$\ln C_t \sim N\left(\ln C_0 + \left[r + m(\mu - r) - \frac{1}{2}m^2\sigma^2\right] \cdot t, m^2\sigma^2 t\right).$$

With respect to the derivation of the mean and variance of the value of the CPPI portfolio V_T^{CPPI} at the end of the investment horizon T, we recall that the mean and the variance of a lognormally distributed random variable $lnX \sim N(\mu, \sigma^2)$ are given by[15]

$$\mu\left(X\right) = E[X] = e^{\mu + \frac{\sigma^2}{2}} \qquad (23)$$

$$\sigma^2\left(X\right) = Var[X] = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right). \qquad (24)$$

Thus, following from the lognormal distribution of the value of the cushion C_T and the final portfolio value of the CPPI strategy (9)

$$V_T^{CPPI} = \alpha_T \cdot V_0^{CPPI} + C_T,$$

we obtain

$$\begin{aligned}\mu\left(V_T^{CPPI}\right) &= E\left[V_T^{CPPI}\right] = \alpha_T \cdot V_0^{CPPI} + E\left[C_T\right] \\ &= \alpha_T \cdot V_0^{CPPI} + e^{\ln C_0 + \left[r+m(\mu-r)-\frac{1}{2}m^2\sigma^2\right]T + \frac{m^2\sigma^2 T}{2}} \\ &= \alpha_T \cdot V_0^{CPPI} + C_0 \cdot e^{[r+m(\mu-r)]T},\end{aligned}$$

and

$$\begin{aligned}\sigma^2\left(V_T^{CPPI}\right) &= Var\left[\alpha_T \cdot V_0^{CPPI} + C_T\right] = Var\left[C_T\right] \\ &= e^{2\cdot\left\{\ln C_0 + \left[r+m(\mu-r)-\frac{1}{2}m^2\sigma^2\right]\cdot T\right\} + m^2\sigma^2 T} \cdot \left(e^{m^2\sigma^2 T} - 1\right) \\ &= C_0^2 \cdot e^{2[r+m(\mu-r)]T} \cdot \left(e^{m^2\sigma^2 T} - 1\right).\end{aligned}$$

[15] See, e.g., Fahrmeir (2003), p.299.

B Calculation of the mean and variance of the OBPI value

Recall the terminal portfolio value of an OBPI strategy at maturity T

$$\begin{aligned} V_T^{OBPI} &= \max\{S_T, X\} - Put_0 \cdot e^{rT} \\ &= \max\{S_T - X, 0\} + X - Put_0 \cdot e^{rT}. \end{aligned}$$

Thus,

$$\mu\left(V_T^{OBPI}\right) = E\left[V_T^{OBPI}\right] = E\left[\max\{S_T - X, 0\}\right] + X - Put_0 \cdot e^{rT}.$$

Substituting the definition of the upper partial moment (15) and Equation (13)

$$X = Put_0 \cdot e^{rT} + \alpha_T \cdot V_0^{OBPI}$$

for the strike price X, we obtain

$$\begin{aligned} E\left[V_T^{OBPI}\right] &= UPM_1\left(S_T, X\right) + \alpha_T \cdot V_0^{OBPI} \\ &= e^{\mu T} Call\left(S_0, X, \mu, \sigma, 0, T\right) + \alpha_T \cdot V_0^{OBPI}. \end{aligned}$$

In order to calculate the variance of the terminal portfolio value $\sigma^2\left(V_T^{OBPI}\right)$ we use the common formula

$$\sigma^2\left(V_T^{OBPI}\right) = E\left[\left(V_T^{OBPI}\right)^2\right] - \left(E\left[V_T^{OBPI}\right]\right)^2.$$

This leads to

$$\begin{aligned} \sigma^2\left(V_T^{OBPI}\right) &= E\left[\left(\max\{S_T - X, 0\} + \alpha_T \cdot V_0^{OBPI}\right)^2\right] - \left(E\left[V_T^{OBPI}\right]\right)^2 \\ &= E\left[\left(\max\{S_T - X, 0\}\right)^2\right] + 2 \cdot \alpha_T \cdot V_0^{OBPI} \cdot UPM_1\left(S_T, X\right) \\ &\quad + \left(\alpha_T \cdot V_0^{OBPI}\right)^2 - UPM_1\left(S_T, X\right)^2 - \left(\alpha_T \cdot V_0^{OBPI}\right)^2 \\ &\quad - 2 \cdot \alpha_T \cdot V_0^{OBPI} \cdot UPM_1\left(S_T, X\right) \\ &= UPM_2\left(S_T, X\right) - UPM_1\left(S_T, X\right)^2 \\ &= S_0^2 e^{2\mu T + \sigma^2 T} \Phi\left(d_1^* + \sigma\sqrt{T}\right) - 2 X S_0 e^{\mu T} \Phi\left(d_1^*\right) + X^2 \Phi\left(d_2^*\right) \\ &\quad - e^{2\mu T} Call\left(S_0, X, \mu, \sigma, 0, T\right)^2, \end{aligned}$$

where

$$\begin{aligned} d_1^* &= \frac{\ln\left(\frac{S_0}{X}\right) + \left(\mu + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}, \\ d_2^* &= d_1^* - \sigma\sqrt{T}. \end{aligned}$$

C Second-order stochastic dominance

C.1 Proof of Theorem 2

Recall the expected values of the two portfolio insurance strategies at maturity T (10) and (16)

$$\begin{aligned} E\left[V_T^{CPPI}\right] &= \alpha_T \cdot V_0^{CPPI} + V_0^{CPPI} \cdot \left(1 - \alpha_T \cdot e^{-rT}\right) \cdot e^{[r+m\cdot(\mu-r)]\cdot T}, \\ E\left[V_T^{OBPI}\right] &= \alpha_T \cdot V_0^{OBPI} + E\left[\max\{S_T - X, 0\}\right] \\ &= \alpha_T \cdot V_0^{OBPI} + e^{\mu T} \cdot Call\left(S_0, X, \mu, \sigma, 0, T\right), \end{aligned}$$

where $S_0 = V_0^{CPPI} = V_0^{OBPI}$ and

$$X = Put_0 \cdot e^{rT} + \alpha_T \cdot S_0, \quad Put_0 = Put_0\left(S_0, X, r, \sigma^i, 0, T\right).$$

Hence,

$$\begin{aligned} E\left[V_T^{OBPI}\right] &\leq E\left[V_T^{CPPI}\right] \\ &\Leftrightarrow \\ e^{\mu T} \cdot Call\left(S_0, X, \mu, \sigma, 0, T\right) &\leq V_0^{CPPI} \cdot \left(1 - \alpha_T \cdot e^{-rT}\right) \cdot e^{\mu T} \cdot e^{(m-1)(\mu-r)T} \\ &\Leftrightarrow \\ Call\left(S_0, X, \mu, \sigma, 0, T\right) &\leq \left(V_0^{CPPI} - \underbrace{V_0^{CPPI} \cdot \alpha_T \cdot e^{-rT}}_{X\cdot e^{-rT} - Put_0}\right) \cdot e^{(m-1)(\mu-r)T} \\ &\overset{put-call-parity}{\Leftrightarrow} \\ Call\left(S_0, X, \mu, \sigma, 0, T\right) &\leq Call\left(S_0, X, r, \sigma^i, 0, T\right) \cdot e^{(m-1)(\mu-r)T}. \end{aligned}$$

C.2 Proof of Theorem 3

Recall the set of real functions with exactly two changes of sign

$$\mathbb{S}_2 = \left\{ \begin{array}{l} H : \mathbb{R} \to \mathbb{R} : \exists s_1, s_2 \in \mathbb{R}, \\ \text{where } H(s) \begin{cases} \geq 0, & s \in (-\infty, s_1) \\ \leq 0, & s \in (s_1, s_2) \\ \geq 0, & s \in (s_2, \infty) \end{cases}, \quad H \neq 0 \end{array} \right\}.$$

The cumulative distribution functions of the two portfolio insurance strategies under consideration are defined as follows, where $V_0 := S_0 = V_0^{CPPI} = V_0^{OBPI}$, $x \in \mathbb{R}$

$$\begin{aligned} F_{V_T^{OBPI}}(x) &= Q\left(\underbrace{\alpha_T \cdot V_0^{OBPI}}_{=:a>0} + \max\{S_T - X, 0\} \leq x\right) \\ &= Q\left(a + (S_T - X)^+ \leq x\right), \end{aligned}$$

with

$$(s-X)^{+}=\begin{cases}0, & s\leq X\\ s-x, & s>X\end{cases}$$

and

$$\begin{aligned}F_{V_T^{CPPI}}(x)&=Q\left(V_T^{CPPI}\leq x\right)\\&=Q\left(\underbrace{\alpha_T\cdot V_0^{CPPI}}_{=a}+\underbrace{C_0\cdot S_0^{-m}\cdot e^{(1-m)\left(r+\frac{1}{2}m\sigma^2\right)T}}_{=:b>0}\cdot S_T^m\leq x\right)\\&=Q\left(a+b\cdot S_T^m\leq x\right).\end{aligned}$$

Our goal is to prove that $H(x)=F_{V_T^{OBPI}}(x)-F_{V_T^{CPPI}}(x)\in\mathbb{S}_2$. Therefore, we have to find the two points x_1,x_2 where the sign of the function H changes, i.e. the intersection points of the cumulative distribution functions $F_{V_T^{OBPI}}$ and $F_{V_T^{CPPI}}$. Notice, that the asset price S is always positive under the assumption of a geometric Brownian motion as underlying stochastic dynamics.

Let $x-a\leq 0$. Since $a,b,S_T>0$, we conclude that

$$F_{V_T^{OBPI}}(x)=Q\left((S_T-X)^{+}\leq x-a\right)=\begin{cases}0, & x-a<0\\ Q\left(S_T\leq X\right), & x-a=0\end{cases},$$

and

$$F_{V_T^{CPPI}}(x)=Q\left(b\cdot S_T^m\leq x-a\right)=0.$$

Hence,

$$H(x)=F_{V_T^{OBPI}}(x)-F_{V_T^{CPPI}}(x)\geq 0,\quad \text{if } x-a\leq 0,$$

which implies that the function H does not change its sign in $(-\infty,a]$. In order for H to be in $\mathbb{S}_2$, it remains to show, that the function changes exactly twice its sign in $(a,+\infty)$. Consequently, we are looking for the zeros of H, i.e. the intersection points of the cumulative distribution functions $F_{V_T^{OBPI}}$ and $F_{V_T^{CPPI}}$, respectively. This leads for $x>a$ to the condition

$$F_{V_T^{OBPI}}(x)=Q\left((S_T-X)^{+}\leq x-a\right)\stackrel{!}{=}Q\left(b\cdot S_T^m\leq x-a\right)=F_{V_T^{CPPI}}(x). \tag{25}$$

For our further calculations, we define for $s>0$

$$f_{V_T^{OBPI}}(s):=(s-X)^{+}\ \text{ and }\ f_{V_T^{CPPI}}(s):=b\cdot s^m,\quad m\geq 1.$$

These are the payoff functions of the two portfolio insurance strategies reduced by the minimum guaranteed terminal portfolio value $\alpha_T\cdot V_0$. Notice, that the inverse functions of $f_{V_T^{OBPI}}$ and $f_{V_T^{CPPI}}$ both exist in $\mathbb{R}^+$. Now, let $x>a$. Substituting $f_{V_T^{OBPI}}$ and $f_{V_T^{CPPI}}$ in Condition (25) leads to

$$\begin{aligned}&F_{V_T^{OBPI}}(x)\stackrel{!}{=}F_{V_T^{CPPI}}(x)\\ \Leftrightarrow\ &Q\left(f_{V_T^{OBPI}}(S_T)\leq x-a\right)=Q\left(f_{V_T^{CPPI}}(S_T)\leq x-a\right)\end{aligned}$$

which is equivalent to

$$f^{-1}_{V_T^{OBPI}}(x-a) = f^{-1}_{V_T^{CPPI}}(x-a).$$

Thus, using $s := f^{-1}_{V_T^{OBPI}}(x-a)$,

$$F_{V_T^{OBPI}}(x) = F_{V_T^{CPPI}}(x)$$

is equivalent to

$$s = f^{-1}_{V_T^{CPPI}}\left(f_{V_T^{OBPI}}(s)\right) \Leftrightarrow f_{V_T^{CPPI}}(s) = f_{V_T^{OBPI}}(s).$$

Hence, the cumulative distribution functions $F_{V_T^{OBPI}}$ and $F_{V_T^{CPPI}}$ of the two investment strategies intersect each other, iff the corresponding payoff functions intersect.[16] In order to show $H \in \mathbb{S}_2$, we therefore have to determine the interception points of $f_{V_T^{OBPI}}$ and $f_{V_T^{CPPI}}$. This leads to

$$f_{V_T^{OBPI}}(s) \overset{!}{=} f_{V_T^{CPPI}}(s) \Leftrightarrow (s-X)^+ = b \cdot s^m,$$

which is equivalent to

$$\begin{aligned} 0 &= b \cdot s^m, && \text{if } 0 < s \leq X \\ s - X &= b \cdot s^m, && \text{if } X < s. \end{aligned}$$

Hence, the two payoff functions do not intersect for $0 < s \leq X$. In fact,

$$f_{V_T^{OBPI}}(s) < f_{V_T^{CPPI}}(s), \text{ if } 0 < s \leq X.$$

To conclude for the case $s > X$, we define the function

$$\begin{aligned} h(s) &= f_{V_T^{CPPI}}(s) - f_{V_T^{OBPI}}(s) \\ &= bs^m - s + X, \end{aligned}$$

and search for zeros of this function in order to find the intersection points of the two payoffs $f_{V_T^{OBPI}}$ and $f_{V_T^{CPPI}}$. Notice, that $h(s) > 0$, if $0 < s \leq X$. Therefore, we try to find parameter restrictions such that the polynomial function h possesses exactly one strictly negative minimum. Then, as h is continuous and diverges to $+\infty$ for $s \to +\infty$, we could conclude that there exist exactly two nulls of h and thus, exactly two intersection points of the payoff functions $f_{V_T^{OBPI}}$ and $f_{V_T^{CPPI}}$, respectively for $s > X$.

[16] Since $f_{V_T^{OBPI}}$ and $f_{V_T^{CPPI}}$ are both strictly monotone increasing for $x > a$, it furthermore holds that

$$F_{V_T^{OBPI}}(x) \lesseqqgtr F_{V_T^{CPPI}}(x) \Leftrightarrow f_{V_T^{CPPI}}(s) \lesseqqgtr f_{V_T^{OBPI}}(s).$$

It holds for $s > X$

$$h'(s) = m \cdot bs^{m-1} - 1 \overset{!}{=} 0 \Leftrightarrow s^* = \left(\frac{1}{bm}\right)^{\frac{1}{m-1}},$$
$$h''(s) = m \cdot (m-1) \cdot \underbrace{b}_{>0} \cdot s^{m-2} > 0, \quad \forall m > 1.$$

Thus, for all $m > 1$ the only extremum s^* is a minimum ($h'' > 0$) with value

$$h(s^*) = b\left(\frac{1}{bm}\right)^{\frac{m}{m-1}} - \left(\frac{1}{bm}\right)^{\frac{1}{m-1}} + X = (bm)^{\frac{1}{1-m}} \cdot \frac{(1-m)}{m} + X.$$

In order to force the function value at the minimum to be negative, the following restriction must be satisfied:

$$h(s^*) \overset{!}{<} 0 \overset{m>1}{\Leftrightarrow} (bm)^{\frac{1}{1-m}} > \frac{m}{m-1} \cdot X.$$

Hence, if $(bm)^{\frac{1}{1-m}} > \frac{m}{m-1} \cdot X, \quad m > 1$, the function h has exactly two zeros s_1, s_2 and the two payoff functions $f_{V_T^{OBPI}}$ and $f_{V_T^{CPPI}}$ intersect exactly two times. We therefore set

$$s_1 := \min\left\{s \in \mathbb{R}^+ : f_{V_T^{OBPI}}(s) = f_{V_T^{CPPI}}(s)\right\},$$
$$s_2 := \max\left\{s \in \mathbb{R}^+ : f_{V_T^{OBPI}}(s) = f_{V_T^{CPPI}}(s)\right\}.$$

If $m = 1$ and $s > X$ there only exists one interception point, i.e.

$$h(s) = bs - s + X \overset{!}{=} 0 \Leftrightarrow s_1 = \frac{X}{1-b}, \quad s_1 > X.$$

Notice, that this point s_1 actually exists, as

$$b = C_0 \cdot S_0^{-m} \cdot e^{(1-m)\left(r+\frac{1}{2}m\sigma^2\right)T} \overset{m=1}{=} 1 - \alpha e^{-rT} < 1.$$

The constraint $\frac{m}{m-1} \cdot X < (bm)^{\frac{1}{1-m}}$, $m > 1$, can be equivalently transformed to

$$\frac{m}{m-1} \cdot X < m^{\frac{1}{1-m}} \cdot \left[C_0 \cdot S_0^{-m} \cdot e^{(1-m)\left(r+\frac{1}{2}m\sigma^2\right)T}\right]^{\frac{1}{1-m}},$$

i.e.

$$\frac{m^{\frac{m}{m-1}}}{m-1} \cdot X < C_0 \cdot \left(1 - \alpha e^{-rT}\right)^{\frac{m}{1-m}} \cdot e^{\left(r+\frac{1}{2}m\sigma^2\right)\cdot T},$$

where $C_0 = S_0 \cdot \left(1 - \alpha e^{-rT}\right)$. Hence,

$$\frac{m}{m-1} \cdot X < (bm)^{\frac{1}{1-m}}, \quad m > 1$$

is equivalent to

$$\frac{1}{m-1} \cdot \left(m \cdot \left(1 - \alpha e^{-rT}\right)\right)^{\frac{m}{m-1}} \cdot X < C_0 \cdot e^{r \cdot T} \cdot e^{\frac{1}{2}m\sigma^2 \cdot T}$$

and thus to

$$\frac{1}{m-1}\cdot\left(m\cdot\left(1-\alpha e^{-rT}\right)\right)^{\frac{m}{m-1}} < \frac{C_0}{X\cdot e^{-r\cdot T}}\cdot e^{\frac{1}{2}\cdot\frac{m}{m-1}\cdot(m-1)\cdot\sigma^2\cdot T}$$

or equivalently

$$\frac{1}{m-1}\cdot\left(\frac{m\cdot\left(1-\alpha e^{-rT}\right)}{e^{\frac{1}{2}(m-1)\sigma^2\cdot T}}\right)^{\frac{m}{m-1}} < \frac{C_0}{X\cdot e^{-r\cdot T}}.$$

Altogether, we have proved for $m=1$ or

$$m>1 \quad \text{and} \quad \frac{1}{m-1}\cdot\left(\frac{m\cdot\left(1-\alpha e^{-rT}\right)}{e^{\frac{1}{2}\cdot(m-1)\sigma^2\cdot T}}\right)^{\frac{m}{m-1}} < \frac{C_0}{X\cdot e^{-r\cdot T}},$$

that

$$\begin{array}{ll} f_{V_T^{OBPI}}(s) < f_{V_T^{CPPI}}(s) & \forall\ s<s_1 \\ f_{V_T^{OBPI}}(s) > f_{V_T^{CPPI}}(s) & \forall\ s_1<s<s_2, \\ f_{V_T^{OBPI}}(s) < f_{V_T^{CPPI}}(s) & \forall\ s_2<s, \end{array}$$

where

$$\begin{array}{l} s_1 := \begin{cases} s_1 = \frac{X}{1-b}, & \text{if } m=1 \\ \min\left\{s\in\mathbb{R}^+ : f_{V_T^{OBPI}}(s) = f_{V_T^{CPPI}}(s)\right\}, & \text{if } m>1 \end{cases} \\ s_2 := \begin{cases} s_2 = +\infty, & \text{if } m=1 \\ \max\left\{s\in\mathbb{R}^+ : f_{V_T^{OBPI}}(s) = f_{V_T^{CPPI}}(s)\right\}, & \text{if } m>1 \end{cases} \end{array}.$$

As mentioned earlier, the zeros of the function h exactly represent the zeros of the function H. Thus for $m=1$ or $\frac{1}{m-1}\cdot\left(\frac{m\cdot\left(1-\alpha e^{-rT}\right)}{e^{\frac{1}{2}\cdot(m-1)\sigma^2\cdot T}}\right)^{\frac{m}{m-1}} < \frac{C_0}{X\cdot e^{-r\cdot T}}$, if $m>1$

$$\begin{array}{ll} H(x)\geq 0 \Leftrightarrow F_{V_T^{OBPI}}(x) \geq F_{V_T^{CPPI}}(x), & \forall x\leq s_1 \\ H(x)\leq 0 \Leftrightarrow F_{V_T^{OBPI}}(x) \leq F_{V_T^{CPPI}}(z), & \forall s_1\leq x\leq s_2 \\ H(x)\geq 0 \Leftrightarrow F_{V_T^{OBPI}}(x) \geq F_{V_T^{CPPI}}(x), & \forall x\geq s_2. \end{array}$$

Hence, $H\in\mathbb{S}_1$, if $m=1$ and $H\in\mathbb{S}_2$, if $m>1$.

D Proof of Theorem 6

Recall the means and variances deduced for the CPPI and the OBPI strategy, where $V_0^{CPPI} = V_0^{OBPI} = V_0 = S_0$ and $C_0 = V_0 \cdot \left(1 - \alpha_T \cdot e^{-r \cdot T}\right)$:

$$E\left[V_T^{CPPI}\right] = \alpha_T \cdot V_0 + V_0 \cdot \left(1 - \alpha_T \cdot e^{-r \cdot T}\right) e^{[r+m(\mu-r)] \cdot T},$$

$$Var\left[V_T^{CPPI}\right] = (V_0)^2 \cdot \left(1 - \alpha_T \cdot e^{-rT}\right)^2 \cdot e^{2 \cdot [r+m(\mu-r)] \cdot T} \cdot \left(e^{m^2 \sigma^2 T} - 1\right),$$

and

$$E\left[V_T^{OBPI}\right] = \alpha_T \cdot V_0 + E\left[\max\left\{S_T - X, 0\right\}\right] = \alpha_T \cdot V_0 + e^{\mu T} Call\left(S_0, \mu, \sigma\right),$$

$$\begin{aligned} Var\left[V_T^{OBPI}\right] &= UPM_2\left(S_T, X\right) - UPM_1^2\left(S_T, X\right)^2 \\ &= S_0^2 \cdot e^{2\mu T + \sigma^2 T} N\left(d_1^* + \sigma\sqrt{T}\right) - 2XS_0 e^{\mu T} N\left(d_1^*\right) \\ &\quad + X^2 N\left(d_2^*\right) - e^{2\mu T} Call^2\left(S_0, \mu, \sigma\right), \end{aligned}$$

where $d_1^* = \frac{\ln\left(\frac{S_0}{X}\right) + \left(\mu + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$, $d_2^* = d_1^* - \sigma\sqrt{T}$. From the translation theorem for the variance we directly conclude that

$$\begin{aligned} E\left[\left(V_T^{CPPI}\right)^2\right] &= Var\left[V_T^{CPPI}\right] + \left(E\left[V_T^{CPPI}\right]\right)^2 \\ &= (S_0)^2 \cdot \left(1 - \alpha_T \cdot e^{-rT}\right)^2 \cdot e^{2 \cdot [r+m(\mu-r)] \cdot T} \cdot e^{m^2 \sigma^2 T} \\ &\quad + \left(\alpha_T \cdot S_0\right)^2 + 2 \cdot \alpha_T \cdot S_0^2 \cdot \left(1 - \alpha_T \cdot e^{-r \cdot T}\right) \cdot e^{[r+m(\mu-r)] \cdot T}. \end{aligned}$$

Recall, that $X = Put_0 \cdot e^{rT} + \alpha S_0$. From put-call-parity follows

$$Call\left(S_0, r, \sigma^i\right) + X \cdot e^{-rT} = \underbrace{Put\left(S_0, r, \sigma^i\right)}_{Put_0} + S_0. \tag{26}$$

Hence,

$$Call\left(S_0, r, \sigma^i\right) = S_0 \cdot \left(1 - \alpha_T \cdot e^{-rT}\right),$$

which leads to

$$\begin{aligned} E\left[\left(V_T^{CPPI}\right)^2\right] &= Call\left(S_0, r, \sigma^i\right)^2 \cdot e^{2 \cdot \left\{[r+m(\mu-r)] \cdot T + \frac{m^2 \sigma^2 T}{2}\right\}} \\ &\quad + 2 \cdot \alpha_T \cdot S_0 \cdot Call\left(S_0, r, \sigma^i\right) \cdot e^{[r+m(\mu-r)] \cdot T} + \left(\alpha_T \cdot S_0\right)^2. \end{aligned}$$

Furthermore,

$$\begin{aligned} E\left[\left(V_T^{OBPI}\right)^2\right] &= Var\left[V_T^{OBPI}\right] + \left(E\left[V_T^{OBPI}\right]\right)^2 \\ &= S_0^2 \cdot e^{2\mu T + \sigma^2 T} \cdot N\left(d_1^* + \sigma\sqrt{T}\right) - 2 \cdot X \cdot S_0 \cdot e^{\mu T} N\left(d_1^*\right) \\ &\quad + X^2 \cdot N\left(d_2^*\right) + \left(\alpha S_0\right)^2 + 2 \cdot \alpha_T \cdot S_0 \cdot e^{\mu T} \cdot Call\left(S_0, \mu, \sigma\right). \end{aligned}$$

From the Black-Scholes formula for the value of a call option $Call\left(S_0, X, \mu, \sigma, 0, T\right)$ we know

$$Call\left(S_0, X, \mu, \sigma, 0, T\right) = S_0 \cdot N\left(d_1^*\right) - X \cdot e^{-\mu \cdot T} \cdot N\left(d_2^*\right).$$

Hence,

$$\begin{aligned} E\left[\left(V_T^{OBPI}\right)^2\right] &= S_0^2 \cdot e^{2\mu T + \sigma^2 T} \cdot N\left(d_1^* + \sigma\sqrt{T}\right) - X \cdot e^{\mu T} \cdot S_0 \cdot N\left(d_1^*\right) \\ &\quad + \left(\alpha S_0\right)^2 + e^{\mu T} \cdot Call\left(S_0, \mu, \sigma\right) \cdot \left(2 \cdot \alpha_T \cdot S_0 - X\right). \end{aligned}$$

Then, $E\left[\left(V_T^{CPPI}\right)^2\right] \leq E\left[\left(V_T^{OBPI}\right)^2\right]$ is true if and only if

$$\begin{aligned} & Call^2\left(S_0, r, \sigma^i\right) \cdot e^{2[r+m(\mu-r)]T + m^2\sigma^2 T} \\ & +2 \cdot \alpha_T \cdot S_0 \cdot Call\left(S_0, r, \sigma^i\right) \cdot e^{[r+m(\mu-r)]T} \\ \leq & \\ & S_0 \cdot e^{2\mu T} \cdot \left[S_0 \cdot e^{\sigma^2 T} \cdot N\left(d_1^* + \sigma\sqrt{T}\right) - X \cdot e^{-\mu T} \cdot N\left(d_1^*\right)\right] \\ & +e^{\mu T} \cdot Call\left(S_0, \mu, \sigma\right) \cdot \left(2\alpha S_0 - X\right). \end{aligned}$$

Setting

$$\begin{aligned} Call\left(S_0 e^{\sigma^2 T}, \mu, \sigma\right) \; : &= Call\left(S_0 e^{\sigma^2 T}, X, \mu, \sigma, 0, T\right) \\ &= S_0 e^{\sigma^2 T} \cdot N\left(d_1^* + \sigma\sqrt{T}\right) - X e^{-\mu \cdot T} \cdot N\left(d_1^*\right), \end{aligned}$$

this is equivalent to

$$\begin{aligned} & e^{\mu T} \cdot Call^2\left(S_0, r, \sigma^i\right) \cdot e^{2\cdot(m-1)\cdot(\mu-r)\cdot T + m^2\sigma^2 T} \\ & +2 \cdot \alpha_T \cdot S_0 \cdot \left[Call\left(S_0, r, \sigma^i\right) \cdot e^{(m-1)(\mu-r)\cdot T} - Call\left(S_0, \mu, \sigma\right)\right] \\ & -S_0 \cdot e^{\mu T} \cdot \left[Call\left(S_0 e^{\sigma^2 T}, \mu, \sigma\right) - Call\left(S_0, \mu, \sigma\right)\right] \\ & +Call\left(S_0, \mu, \sigma\right) \cdot e^{\mu T} \cdot \left(X \cdot e^{-\mu T} - S_0\right) \\ \leq\ & 0. \end{aligned}$$

Setting

$$\varDelta := \varDelta\left(S_0, \mu, \sigma\right) := \frac{Call\left(S_0 e^{\sigma^2 T}, \mu, \sigma\right) - Call\left(S_0, \mu, \sigma\right)}{Call\left(S_0, \mu, \sigma\right)},$$

we conclude that

$$E\left[\left(V_T^{CPPI}\right)^2\right] \leq E\left[\left(V_T^{OBPI}\right)^2\right]$$

is equivalent to

$$\begin{aligned}
& e^{\mu T} \cdot Call^2\left(S_0, r, \sigma^i\right) \cdot e^{2 \cdot (m-1) \cdot (\mu - r) \cdot T + m^2 \sigma^2 T} \\
& +2 \cdot \alpha_T \cdot S_0 \cdot \left[Call\left(S_0, r, \sigma^i\right) \cdot e^{(m-1)(\mu-r) \cdot T} - Call\left(S_0, \mu, \sigma\right) \right] \\
& -S_0 \cdot e^{\mu T} \cdot Call\left(S_0, \mu, \sigma\right) \cdot \Delta \\
& +e^{\mu T} \cdot Call\left(S_0, \mu, \sigma\right) \cdot \left(X \cdot e^{-\mu \cdot T} - S_0\right) \\
\leq\ & 0.
\end{aligned}$$

Setting

$$\begin{aligned}
f(m) := & e^{\mu T} \cdot Call^2\left(S_0, r, \sigma^i\right) \cdot e^{2 \cdot (m-1) \cdot (\mu - r) \cdot T + m^2 \sigma^2 T} \\
& +2 \cdot \alpha_T \cdot S_0 \cdot Call\left(S_0, r, \sigma^i\right) \cdot e^{(m-1)(\mu-r) \cdot T}
\end{aligned}$$

which is strictly monotone increasing in m,

$$E\left[\left(V_T^{CPPI}\right)^2\right] \leq E\left[\left(V_T^{OBPI}\right)^2\right]$$

is equivalent to

$$f(m) \leq Call\left(S_0, \mu, \sigma\right) \cdot \left[S_0 \cdot \left(2 \cdot \alpha_T + e^{\mu T} \cdot (1 + \Delta)\right) - X\right] =: b.$$

Setting

$$m_{\max} := f^{-1}(b),$$

we finally derive the condition

$$E\left[\left(V_T^{CPPI}\right)^2\right] \leq E\left[\left(V_T^{OBPI}\right)^2\right] \Leftrightarrow m \leq m_{\max}.$$

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