# Empirical Evaluation of Hybrid Defaultable Bond Pricing Models

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**Abstract**

We present a four-factor model (the extended model of Schmid and Zagst) for pricing credit risk related instruments such as defaultable bonds or credit derivatives. It is a consequent advancement of our prior three-factor model (see Schmid & Zagst (2000)). In addition to a firm-specific credit risk factor we include a new systematic risk factor in form of the GDP growth rates. We set this new model in the context of other hybrid defaultable bond pricing models and empirically compare it to specific representatives. In analogy to Krishnan, Ritchken & Thomson (2005) we find that a model only based on firm-specific variables is unable to capture changes in credit spreads completely. However, consistent to Krishnan et al. (2005) we show that in our model market variables such as GDP growth rates, non-defaultable interest rates and firm-specific variables together significantly influence credit spread levels and changes.

## 1 Introduction

A great variety of credit risk pricing models have been developed over the last thirty years. These models can be divided into four main categories. The first and second generation structural-form models, the reduced-form models and the hybrid models.

The first generation structural-form models go back to Merton (1974) who basically uses the Black-Scholes framework to model default risk. He assumes that a firm defaults if the market value of the firm falls below the value of the liabilities of the firm. If the firm's liabilities can be entirely represented by a zero-coupon bond, there are two possibilities at the maturity of the debt: If the market value of the firm is greater than the face value of the bond, then the bondholder is completely paid off. Otherwise the bondholder only gets back the market value of the firm. Therefore, the payoff to the bondholder is the

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minimum of the face value of the bond and the market value of the firm. Or to say it somehow different, it is the face value of the bond minus a put option on the value of the firm, with a strike price equal to the face value of the bond and a maturity equal to the maturity of the bond. Following this basic intuition, it is easy to derive an explicit formula for the price of a defaultable zero-coupon bond. Some later models try to get rid of some of the unrealistic model assumptions of Merton's framework. Black & Cox (1976) incorporate classes of senior and junior debt, safety covenants, dividends, and restrictions on cash distributions to shareholders, Geske (1977) considers coupon bonds by using a compound options approach and provides a formula for subordinate debt within this framework, Ho & Singer (1982) allow for different maturities of debt and examine the effect of alternative bond indenture provisions such as financing restriction of the firm, priority rules and payment schedules. Leland (1994) extends the model further to incorporate bankruptcy costs and taxes which makes it possible to work with optimal capital structure.

The second generation structural-form models assume that default may occur any time between issuance and maturity of the debt. Default is triggered as soon as the value of the firm's assets reaches a lower threshold level. Often these models are also called first passage time models. Examples are the models of Kim, Ramaswamy & Sundaresan (1993), Longstaff & Schwartz (1995), Briys & de Varenne (1997), Nielsen, Saà-Requejo & Santa-Clara (1993), and Saá-Requejo & Santa-Clara (1997). For further details on these approaches see, e.g., Schmid (2004).

Despite the immense efforts devoted to generalize Merton's methodology, all of these structural models have only limited success in explaining the behavior of prices of debt instruments and credit spreads. It is well documented that structural models which are calibrated to histroical default rates and restricted to reasonable risk-premia specifications generate unrealistically low yield spreads for investment-grade debt, especially for debt of short maturities (see, e.g., Huang & Huang (2003) and Eom, Helwege & Huang (2004)). In addition, Eom et al. (2004) show that the predictive power of structural models is very poor. They find that the mean absolute errors in spreads are more than 70% of the true spread. Jones, Mason & Rosenfeld (1984) come to the same result. Most of the structural models predict a humped shape for the term structure of credit spreads. But Litterman & Iben (1991) find evidence for increasing spread structures, Sarig & Warga (1989) for negative slope and Helwege & Turner (1999) for positive slope in case of speculative grade debt. Lardic & Rouzeau (n.d.) investigate whether structural models are able to reproduce the risk ranking of obligors using corporate bond data of French firms. They find that the theoretical prices do not reproduce the risk ranking in the market. But at least they can show that the structural models are able to track changes in the credit quality of obligors.

The problems of the structural approaches have led to attempts to use models that make more direct assumptions about the default process. These alternative approaches, called reduced-form models, don't consider the relation between default and asset value in an explicit way but model default as a stopping time of

some given hazard rate process, i.e. the default process is specified exogenously. This achieves two effects. The first is that the model can be applied to situations where the underlying asset value is not observable. Second, the default time is unpredictable and therefore the behavior of credit spreads for short maturities can be captured more realistically. In addition, the approach is very tractable and flexible to fit the observed credit spreads. The family of reduced-form models originated with Jarrow & Turnbull (1992). Since that time a long list of papers has appeared which follow this approach. Some of the most important ones are the following. Jarrow & Turnbull (1995) present a model where default is driven by a Poisson process with a constant intensity parameter and a given payoff at default. In Lando (1994), Lando (1997), and Lando (1998) default is driven by Cox processes, which can be thought of as Poisson processes with random intensity parameters. Duffie & Singleton (1999) and Duffie & Singleton (2003) show that valuation under the risk-adjusted probability measure can be executed by discounting the non-defaultable payoff on the debt by a discount rate that is adjusted for the parameters of the default process. Schönbucher (1996) presents a generalization in a Heath-Jarrow-Morton framework[1] that allows for restructuring of defaulted debt and multiple defaults.
Empirical evidence concerning reduced-form models is rather limited. Duffee (1999) finds that these models have difficulty in explaining the observed term structure of credit spreads across firms of different credit risk qualities. Düllmann & Windfuhr (2000) consider specific Vasicek and CIR type reduced form models and show that both models fail to account for all observed shapes of the credit spread structure.

There is a heated debate which class of models - structural or reduced-form - is best (see, e.g., Jarrow, Deventer & Wang (2003)). Jarrow & Protter (2004) compare structural and reduced-form models from an information based perspective. They claim that the models are basically the same - the only distinction between the two model types is not whether the default time is predictable or inaccessible, but whether the information is observed by the market or not. Structural models assume that the modeler has the same complete information as the firm's manager, whereas reduced-form models assume that the modeler has the same incomplete information as the market. They conclude that for pricing and hedging, reduced form models are the preferred methodology.

Hybrid models try to combine ideas of structural and reduced-form models and to get rid of their drawbacks. Basically these models can be seen as a variant of reduced-form modeling with state variables. The conditional probability of default is directly related to specific macro- and/or microeconomic factors. Microeconomic factors can contain firm specific structural information. Important examples are Madan & Unal (1998), Cathcart & El-Jahel (1998), Davydov, Linetsky & Lotz (1999), Madan & Unal (2000), Schmid & Zagst (2000), Bakshi, Madan & Zhang (2006), and Jarrow & Yildirim (2002). Some of these authors

[1] For an introduction to the Heath-Jarrow-Morton framework see, e.g., Zagst (2002).

have parameterized the instataneous credit spread as a function, usually affine, of candidate economic and firm-specific state variables and then directly estimated the effects of these variables. E.g., one of the factors that determine the credit spread in the model of Schmid & Zagst (2000) is the so called uncertainty index which can be understood as an aggregation of all information on the quality of the firm currently available: The greater the value of the uncertainty process the lower the quality of the firm. A similar idea to this uncertainty process was first introduced by Cathcart & El-Jahel (1998). In their model default is explicitly driven by a so called signaling process which is assumed to follow a geometric Brownian motion. The second underlying process is the non-defaultable short rate which is assumed to follow a mean reverting square root process. The model of Schmid & Zagst (2000) differs from Cathcart and El-Jahel in several ways: First, they assume the underlying non-defaultable short rate to follow either a mean reverting Hull-White process or a mean-reverting square root process with time-dependent mean reversion level. The uncertainty or signaling process is assumed to follow a mean reverting square root process. Finally, in addition to the non-defaultable short rate and the uncertainty index, they directly model the short rate spread process: they assume that the spread between a defaultable and a non-defaultable bond is considerably driven by the uncertainty index but that there may be additional factors which influence the level of the spreads: at least the contractual provisions, liquidity and the premium demanded in the market for similar instruments have a great impact on credit spreads. The model of Schmid & Zagst (2000) was empirically tested by Schmid & Kalemanova (2002) for German and Italian sovereign bond data over a period of two years.

Based on these encouraging results for time histories up to two years we want to further improve the model to make it more stable even for long-term horizons of ten years and more. Therefore, we extend the original three-factor approach to a model where we directly consider the general state of the economy in terms of GDP growth rates. We assume that the general state of the economy is an important factor for explaining interest rate and spread levels and modeling systematic risks. In addition, we keep the uncertainty index to model the firm-specific default risk. We show that the additional factor in our model has a significant impact on the quality of the model. By empirically comparing the new model with other hybrid models we close a gap in the literature: So far there are no empirical analyses systematically evaluating and comparing different hybrid models. In different models the conditional probability of default and the credit spread are directly related to specific macro- and/or microeconomic factors. We consider three specific representatives: The model of Schmid & Zagst (2000) as an example for the group of models where the spread is dependent on a firm-specific factor. The model of Bakshi et al. (2006) which belongs to the group of models where the spread depends on non-defaultable interest rates and a firm-specific factor and finally the extended model of Schmid and Zagst who make the spread dependent on the general state of the economy and a firm-specific factor. In analogy to Krishnan et al. (2005) we find that a model only based on firm-specific variables is unable to capture changes in credit spreads completely. However, consistent to Krishnan et al. (2005) we show that market variables such as GDP growth rates, non-defaultable interest rates and firm-specific variables together significantly influence credit spread levels and changes.

The rest of the paper is organized as follows. In Section 2 we present the general market model and the data we use to test the different hybrid defaultable bond pricing models. We consider only models that parameterize credit spreads not only as a function of interest rates but also of firm-specific information. Therefore, in Sections 3, 4 and 5 we give an overview of the model of Schmid & Zagst (2000), the extended model of Schmid and Zagst and the approach of Bakshi et al. (2006), respectively. For each model, we show the closed form pricing formulas for defaultable and non-defaultable bonds and explain their main differences, especially with regards to their structure. In addition, we estimate the parameters of all processes using Kalman filter methods as suggested by Schmid (2004). Finally, Section 6 shows the results of empirically comparing the three models. We compare the performance of the models by applying different in- and out-of sample tests. We calculate the average absolute pricing errors and the average coefficients of determination $R^2$ to measure how well the estimated changes in the defaultable and non-defaultable zero rates can describe the actually observed rates. We close in Section 7 with a short summary of our main findings.

# 2 Market Model and Data

In the following, we assume that markets are frictionless and perfectly competitive, that trading takes place continuously, that there are no taxes, transaction costs, or informational asymmetries, and that investors act as price takers. We fix a terminal time horizon $T^*$. Uncertainty in the financial market is modeled by a probability space $(\Omega, \mathcal{G}, \mathbf{P})$, and therefore all random variables and stochastic processes introduced below are defined on this probability space. We assume that $(\Omega, \mathcal{G}, \mathbf{P})$ is equipped with three filtrations $\mathbb{H}$, $\mathbb{F}$, and $\mathbb{G}$, i.e. three increasing and right-continuous families of sub-$\sigma$-fields of $\mathcal{G}$.

The default time $T^d$ of an obligor is an arbitrary non-negative random variable on $(\Omega, \mathcal{G}, \mathbf{P})$. For the sake of convenience we assume that $\mathbf{P}\left(T^d = 0\right) = 0$ and $\mathbf{P}\left(T^d > t\right) > 0$ for every $t \in (0, T^*]$. For a given default time $T^d$, we introduce the associated default indicator function as $H(t) = \mathbf{1}_{\{T^d \leq t\}}$ and the survival indicator function as $L(t) = 1 - H(t), t \in (0, T^*]$. Let $\mathbb{H} = (\mathcal{H}_t)_{0 \leq t \leq T^*}$ be the filtration generated by the process $H$. In addition, we define the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$ as the filtration generated by the multi-dimensional standard Brownian motion $W^{'}(t) = (W_r(t), W_w(t), W_u(t), W_s(t))$ and $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T^*}$ as the enlarged filtration $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, i.e. for every $t$ we set $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$. All filtrations are assumed to satisfy the usual conditions of completeness and right-continuity. For the sake of simplicity we furthermore assume that $\mathcal{F}_0$ is trivial.

Further, we assume that there exist two $\mathbb{F}$-adapted processes, the short-term interest rate process $r(t)$ and the short-term spread process $s(t)$. It should be emphasized that $T^d$ is not necessarily a stopping time with respect to the filtration $\mathbb{F}$ but of course with respect to the filtration $\mathbb{G}$. If we assumed that $T^d$ was a stopping time with $\mathbb{F}$, then it would be necessarily a predictable stopping time. This situation is the case in all traditional structural models.

We will assume throughout that for any $t \in (0, T^*]$ the $\sigma-$fields $\mathcal{F}_{T^*}$ and $\mathcal{H}_t$ are conditionally independent (under the martingale measure $\mathbf{Q}$) given $\mathcal{F}_t$. This is equivalent to the assumption that $\mathbb{F}$ has the so-called martingale invariance property with respect to $\mathbb{G}$, i.e. any $\mathbb{F}-$martingale follows also a $\mathbb{G}-$martingale (see Bielecki and Rutkowski (2004), p. 167). For the technical proofs we will use another condition which is also known to be equivalent to the martingale invariance property (see Bielecki and Rutkowski (2004), p. 242): For any $t \in (0, T^*]$ and any $\mathbf{Q}-$integrable $\mathcal{F}_{T^*}-$measurable random variable $X$ we have $E^{\mathbf{Q}}[X|\ \mathcal{G}_t] = E^{\mathbf{Q}}[X|\ \mathcal{F}_t]$. The modeling takes place after measure transformation, i.e. we assume that there exists a measure $\mathbf{Q} \sim \mathbf{P}$ such that all discounted price processes of the financial instruments in our market are $\mathbf{Q}$-martingales with respect to a suitable numéraire.

We can invest in the following instruments:

- The non-defaultable money market account defined by

$$B(t) = e^{\int_0^t r(l)dl},$$

  which we take as numéraire.

- Non-defaultable zero-coupon bonds with face value 1 and maturities $T \in [0, T^*]$, whose price processes are denoted by $(P(t,T))_{0 \le t \le T}$.

- Defaultable zero-coupon bonds with face value 1 and maturities $T \in [0, T^*]$ that pay 1 at maturity $T$, if there has been no default before time $T$, and the recovery rate $z\left(T^d\right)$ (expressed as fraction of the market value $P^d_-(T^d, T)$ of the bond just prior to default) at default $T^d$, if $T^d \le T$. We assume that $z\left(t\right)$ is a $\mathcal{F}_t$-adapted, continuous process with $z\left(t\right) \in [0, 1)$ for all $t$ and denote the price processes of defaultable zero-coupon bonds by $\left(P^d(t,T)\right)_{0 \le t \le T}$.

- The defaultable money market account which is defined by

$$B^d(t) = \left(1 + \int_0^t (z(l) - 1)dH(l)\right) e^{\int_0^t r(l)+s(l)L(l)dl}.$$

Under the measure $\mathbf{Q}$ the prices of the financial instruments can be calculated as conditional expected values of the discounted future payoffs, i.e.

$$P\left(t,T\right) = \mathbb{E}^{\mathbf{Q}}\left(e^{-\int_t^T r(l)dl}\middle|\ \mathcal{F}_t\right) \tag{1}$$

and for $t < \min(T^d, T)$

$$\begin{aligned} & P^d(t,T) \\ = \ & \mathbb{E}^{\mathbf{Q}} \left( \int_t^T e^{-\int_t^u r(l)dl} z(u) P^d_-(u,T) dH(u) + e^{-\int_t^T r(l)dl} L(T) \middle| \mathcal{G}_t \right). \end{aligned}$$

Given some technical integrability conditions for $r$ and $s$ (see, e.g., Schmid (2004)), it can be shown that for $t < \min(T^d, T)$, $P^d(t,T)$ can be expressed as

$$P^d(t,T) = \mathbb{E}^{\mathbf{Q}} \left[ e^{-\int_t^T (r(l)+s(l))dl} \middle| \mathcal{G}_t \right]. \tag{2}$$

In the following sections we show three different examples how to choose $r$ and $s$. First we introduce the original three-factor model of Schmid & Zagst (2000), then we show their extended four-factor model and finally we give an overview of the model of Bakshi et al. (2006). We determine the pricing formulas of defaultable zero-coupon bonds in the three approaches, estimate their parameters based on the same data set and empirically compare the performance of the three models.

To empirically compare different defaultable term-structure models we use time series of defaultable and non-defaultable zero rates for maturities of 1 year, 2, 3, 4, 5, 7, and 10 years. As our main data source we use Bloomberg. All prices are in US dollars so that we do not have to deal with currency risks at all. For the non-defaultable zero rates we use US Treasury Strips (see Figure 1). For defaultable zero rates we consider average rates of American Industrials of the two rating classes $A2$ and $BBB1$ (see Figures 2 and 3). The high spread levels in 1998 may be explained by the Asian and Russian crisis, and the increase in 2001 by September 11 and the market crash of $2001-02$. We use weekly data from October 1, 1993, until December 31, 2004, which we transform into continuous zero rates and smooth by means of Nelson-Siegel curves. These are 588 observations. For parameter estimations we use the first 401 data points, the remaining data we use for out-of-sample tests. All parameters are estimated using Kalman filter methodologies as suggested, e.g., by Schmid (2004). In addition to the interest rate data, we consider quarterly growth rates of the US GDP. As there is neither monthly nor weekly data observable we generate weekly from quarterly data by linear interpolation taking into account a delay of 3 month due to publication. For all our estimations we use the software package S-PLUS finmetrics.

# 3 The Model of Schmid and Zagst

In the following section we give a short overview of the Schmid and Zagst three-factor defaultable term structure model. As a typical hybrid model it

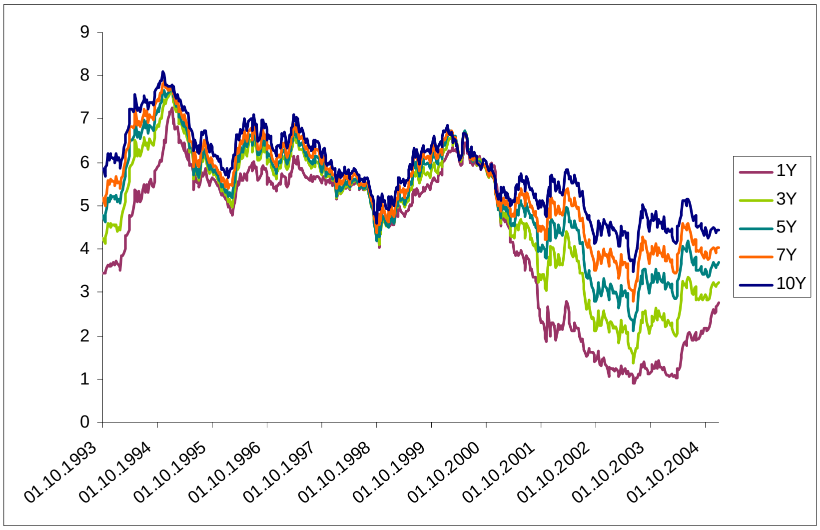


Figure 1: 1- to 10-Year Treasury Strips in %. Time Period: 1993 - 2004. Source: Bloomberg.

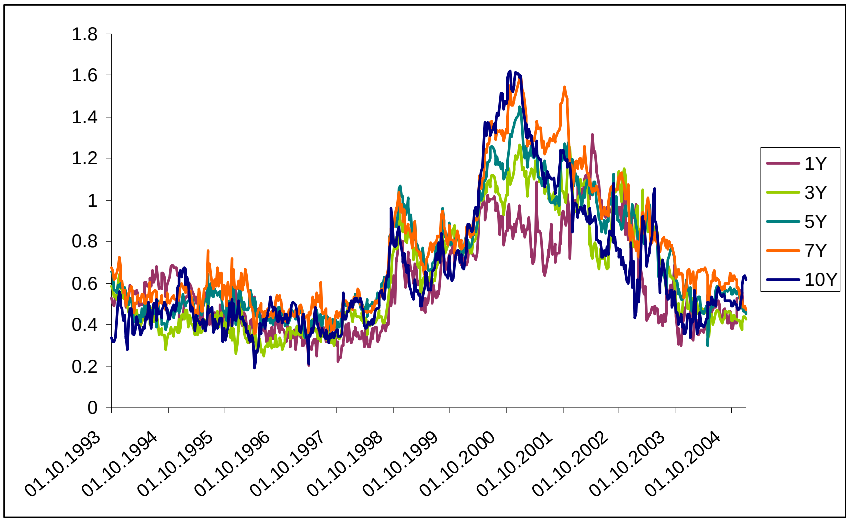


Figure 2: 1- To 10-Year Credit Spreads of US Industrials A2 in %. Time Period: 1993 – 2004. Source: Bloomberg.

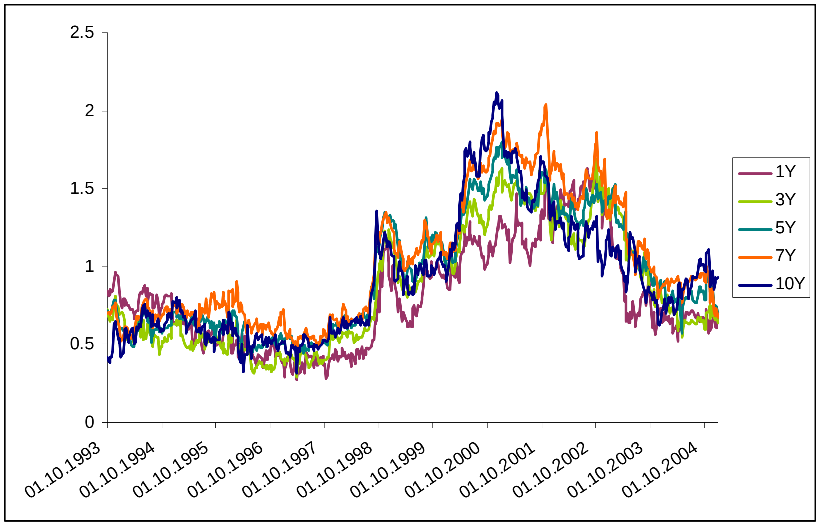


Figure 3: 1- To 10-Year Credit Spreads of US Industrials BBB1 in %. Time Period: 1993 – 2004. Source: Bloomberg.

combines elements of structural and reduced-form models. The underlying non-defaultable short rate is assumed to either follow a mean reverting Hull-White process or a mean-reverting square root process with time-dependent mean reversion level. Therefore, the dynamics of the non-defaultable short rate are given by the following stochastic differential equation (SDE):

$$dr(t) = [\theta_r(t) - a_r r(t)]\, dt + \sigma_r r(t)^{\beta}\, dW_r(t), \; 0 \le t \le T^*, \tag{3}$$

where $a_r$, $\sigma_r > 0$ are positive constants, $\beta = 0$ or $\frac{1}{2}$, and $\theta_r$ is a non–negative valued deterministic function. This specification implies that the current rate $r(t)$ is pulled towards $\frac{\theta_r(t)}{a_r}$ with a speed of adjustment $a_r$, and if $\beta = \frac{1}{2}$ the instantaneous variance of the change in the rate is proportional to its level.

One of the factors that determine the credit spread is the so called uncertainty index which can be understood as an aggregation of all information on the quality of the firm currently available: The greater the value of the uncertainty process the lower the quality of the firm. The uncertainty or signaling process is assumed to follow a mean reverting square root process. The development of the uncertainty index is given by the following stochastic differential equation:

$$du(t) = [\theta_u - a_u u(t)]\, dt + \sigma_u \sqrt{u(t)} dW_u(t), \; 0 \le t \le T^*, \tag{4}$$

where $a_u$, $\sigma_u > 0$ are positive constants and $\theta_u$ is a non–negative constant.

The dynamics of the short rate spread (the short rate spread is supposed to be the defaultable short rate minus the non-defaultable short rate) is given by the following stochastic differential equation:

$$ds(t) = [b_s u(t) - a_s s(t)]\,dt + \sigma_s \sqrt{s(t)} dW_s(t), \; 0 \le t \le T^*, \tag{5}$$

where $a_s$, $b_s, \sigma_s > 0$ are positive constants. Therefore, the uncertainty index has a great impact on the mean reversion level of the short rate spread.

Additionally, it is assumed that

$$\begin{aligned} Cov(dW_r(t), dW_s(t)) &= Cov(dW_r(t), dW_u(t)) \\ &= Cov(dW_s(t), dW_u(t)) = 0. \end{aligned}$$

Although Schmid and Zagst assume uncorrelated standard Brownian motions $W_r$, $W_s$, and $W_u$, the short rate spread $s(t)$ and the uncertainty index $u(t)$ are correlated through the stochastic differential equation for the short rate spread. Note, that the system of stochastic differential equations as given by Equations (3) - (5), has a unique strong solution for each given initial value $(r_0, u_0, s_0)' \in \mathbb{R}^3$. If we now define a progressively measurable process $\gamma(t) = (\gamma_r(t), \gamma_u(t), \gamma_s(t))'$ such that

$$\begin{aligned} \gamma_r(t) &= \lambda_r \sigma_r r(t)^{1-\beta}, \\ \gamma_u(t) &= \lambda_u \sigma_u \sqrt{u(t)}, \text{ and} \\ \gamma_s(t) &= \lambda_s \sigma_s \sqrt{s(t)}, \; 0 \le t \le T^*, \end{aligned}$$

for real constants $\lambda_r$, $\lambda_u$, $\lambda_s$, by applying Girsanov's theorem we can show that

$$\widehat{W}(t) = W(t) + \int_0^t \gamma(l)dl$$

is a standard Brownian motion under the measure $\mathbf{Q}$. Then the $\mathbf{Q}$-dynamics of $r$, $s$, and $u$ are given by

$$\begin{aligned} dr(t) &= [\theta_r(t) - \hat{a}_r r(t)]\,dt + \sigma_r r(t)^\beta d\hat{W}_r(t), && (6) \\ ds(t) &= [b_s u(t) - \hat{a}_s s(t)]\,dt + \sigma_s \sqrt{s(t)} d\hat{W}_s(t), && (7) \\ du(t) &= [\theta_u - \hat{a}_u u(t)]\,dt + \sigma_u \sqrt{u(t)} d\hat{W}_u(t), \; 0 \le t \le T^*, && (8) \end{aligned}$$

where $\hat{a}_i = a_i + \lambda_i \sigma_i^2$. Using Equation (1) and Equation (6) we can calculate the price of a non-defaultable zero-coupon bond in the Schmid and Zagst model:

**Theorem 1 (Price of a non-defaultable zero-coupon bond)** *The time $t$ price of a non-defaultable zero-coupon bond with maturity $T$ is given by*

$$P(t,T) = \mathbb{E}^{\mathbf{Q}}\left[ e^{-\int_t^T r(l)dl} \middle| \mathcal{F}_t \right] = P(t,T,r(t)),$$

*where*

$$P(t,T,r) = e^{A(t,T) - B(t,T)r}$$

*with*

$$B(t,T) = \begin{cases} \frac{1}{\hat{a}_r}\left[\,1-e^{-\hat{a}_r(T-t)}\right], & \text{if } \beta=0, \\ \frac{1-e^{-\delta_r(T-t)}}{\kappa_1^{(r)}-\kappa_2^{(r)}e^{-\delta_r(T-t)}}, & \text{if } \beta=\frac{1}{2}, \end{cases} \quad, \textit{ and,} \tag{9}$$

$$\ln A(t,T) = \begin{cases} \int_t^T \left(\frac{1}{2}\sigma_r^2 B(\tau,T)^2 - \theta_r(\tau)\,B(\tau,T)\right)d\tau & \\ = \ln\frac{P(0,T)}{P(0,t)} - B(t,T)\,\frac{\partial \ln P(0,t)}{\partial t} & \text{if } \beta=0, \\ -\frac{\sigma_r^2}{4\hat{a}_r^3}\left(e^{-\hat{a}_rT}-e^{-\hat{a}_rt}\right)^2\left(e^{2\hat{a}_rt}-1\right), & \\ -\int_t^T \theta_r(\tau)\,B(\tau,T)\,d\tau, & \text{if } \beta=\frac{1}{2}, \end{cases} \tag{10}$$

*with*

$$\delta_x = \sqrt{\hat{a}_x^2+2\sigma_x^2} \textit{ and } \kappa_{1/2}^{(x)} = \frac{\hat{a}_x}{2} \pm \frac{1}{2}\delta_x. \tag{11}$$

**Proof.**

See, e.g., Hull & White (1990). ■

Using Equation (2) and Equations (6) - (8) we can calculate the price of a defaultable zero-coupon bond in the Schmid and Zagst model:

**Theorem 2 (Price of a defaultable zero-coupon bond)** *The price of a defaultable zero-coupon bond at time $t<\min(T^d,T)$ is given by*

$$P^d(t,T) = \mathbb{E}^{\mathbf{Q}}\left[e^{-\int_t^T(r(l)+s(l))dl}\,\middle|\,\mathcal{G}_t\right] = P^d(t,T,r(t),s(t),u(t)),$$

*where*

$$P^d(t,T,r,s,u) = e^{A^d(t,T)-B(t,T)r-C^d(t,T)s-D^d(t,T)u} \tag{12}$$

*with*

$$\begin{aligned} C^d(t,T) &= \frac{1-e^{-\delta_s(T-t)}}{\kappa_1^{(s)}-\kappa_2^{(s)}e^{-\delta_s(T-t)}}, \\ D^d(t,T) &= \frac{-2v'(t,T)}{\sigma_u^2 v(t,T)}, \\ \ln A^d(t,T) &= \ln A(t,T) + \frac{2\theta_u}{\sigma_u^2}\ln\left|\frac{v(T,T)}{v(t,T)}\right|, \end{aligned}$$

*where $v(t,T)$ is a complicated function given in the appendix.*

**Proof.**

See, e.g., Schmid & Zagst (2000). ■

If we estimate the parameters (for the case $\beta=0$ and $b_s=1$) using the data specified in Section 2 by application of Kalman filter techniques we get the estimates as specified in Tables 1 and 2.

Table 1: Results of the parameter estimations for the process $r$ in the model of Schmid and Zagst using non-defaultable weekly bond data from October 1, 1993, until June 1, 2001.

| | |
|---|---|
| $a_r$ | 0.14309 |
| $\sigma_r$ | 0.00655 |
| $\hat{a}_r$ | 0.05035 |

Table 2: Results of the parameter estimations for the processes $s$ and $u$ in the model of Schmid and Zagst using defaultable weekly bond data of American Industrials for the rating classes A2, and BBB1 from October 1, 1993, until June 1, 2001.

| | BBB1 | A2 |
|---|---|---|
| $a_s$ | 1.700367 | 1.699441 |
| $\sigma_s$ | 0.059210 | 0.078726 |
| $a_u$ | 0.092899 | 0.154996 |
| $\sigma_u$ | 0.047829 | 0.018979 |
| $\theta_u$ | 0.001181 | 0.001660 |
| $\hat{a}_s$ | 1.353286 | 1.085847 |
| $\hat{a}_u$ | 0.024273 | 0.144247 |

Based on these parameter estimates we can calculate the mean reversion levels of the stochastic processes. For $r$ we get an average mean reversion level of 5.55%. Note, that the average observed 3-month rate is approximately 5.09%. For rating categories $BBB1$ and $A2$ the mean reversion levels of $u$ are 1.27% and 1.07%. For $s$ the $BBB1$ and $A2$ mean reversion levels are 75 bp and 63 bp, respectively. These numbers are rather intuitive as the average 3-month spreads for rating categories $BBB1$ and $A2$ approximately equal 69 bp and 60 bp, respectively.

# 4 The Extended Model of Schmid and Zagst

The extended model of Schmid and Zagst is a four-factor model which considers macro- as well as microeconomic factors:

- For the non-defaultable short rate $r$ a two-factor Hull-White type model is used, i.e. the dynamics of the non-defaultable short rate are given by

$$dr(t) = (\theta_r(t) + b_r w(t) - a_r r(t))\, dt + \sigma_r dW_r(t), \tag{13}$$

$$dw(t) = (\theta_w - a_w w(t))\, dt + \sigma_w dW_w(t). \tag{14}$$

$a_r$, $b_r$, $\sigma_r$, $a_w$, $\sigma_w$, $a_u$ are positive constants, $\theta_w$ is a non-negative constant and $\theta_r$ is a continuous, deterministic function. Unlike in other models $w$ is not assumed to be unobservable, but $w$ is fitted to the GDP growth rate of the previous quarter year. The idea is to directly model the dependence of interest rate levels on general economic conditions.

- The dynamics of the short rate spread are expressed according to

$$\begin{aligned} ds(t) &= (\theta_s + b_{su}u(t) - b_{sw}w(t) - a_s s(t))\,dt + \sigma_s dW_s(t), \quad (15) \\ du(t) &= (\theta_u - a_u u(t))\,dt + \sigma_u dW_u(t). \quad (16) \end{aligned}$$

  $a_r$, $b_r$, $\sigma_r$, $a_w$, $\sigma_w$, $a_u$, $\sigma_u$, $b_{su}$, $b_{sw}$, $a_s$, $\sigma_s$ are positive constants, $\theta_w$, $\theta_u$, $\theta_s$ are non-negative constants, $\theta_r$ is a continuous, deterministic function, and $W = (W_r, W_w, W_u, W_s)'$ is a four-dimensional standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$. The drift of the short rate spread $s$ depends on the two additional factors $u$ and $w$. For the systematic factor $w$ the interpretation is as follows: Spreads usually widen in bearish markets and tighten in bullish markets. E.g., the yearly US GDP growth rates and the default probabilities[2] of $BBB$ rated bonds show a negative correlation of $-0.49$ based on data between 1990 and 2002. Through $w$ we ensure that non-defaultable rates and credit spreads are negatively correlated. This is in analogy to Longstaff & Schwartz (1995) and Duffee (1998) who find that treasury yields are negatively correlated with changes in credit spreads. $u$ is the unceratinty index which incorporates (in addition to the systematic factor $w$) a firm specific (or rating specific) component of default risk to the spread process. This is in line with Krishnan et al. (2005) who emphasize that firm-specific risk variables influence credit spread levels.

Note that the system of stochastic differential equations as given by Equations (13) - (16), has a unique strong solution for each given initial value $(r_0, w_0, u_0, s_0)' \in \mathbb{R}^4$. If we now define a progressively measurable process $\gamma(t) = (\gamma_r(t), \gamma_w(t), \gamma_u(t), \gamma_s(t))'$ such that

$$\begin{aligned} \gamma_r(t) &= \lambda_r \sigma_r r(t), \\ \gamma_w(t) &= \lambda_w \sigma_w w(t), \\ \gamma_u(t) &= \lambda_u \sigma_u u(t), \\ \gamma_s(t) &= \lambda_s \sigma_s s(t), \end{aligned}$$

for real constants $\lambda_r$, $\lambda_w$, $\lambda_u$, $\lambda_s$, by applying Girsanov's theorem we can show that

$$\widehat{W}(t) = W(t) + \int_0^t \gamma(l)dl$$

[2]Source: S & P Rating Transitions

is a standard Brownian motion under the measure $\mathbf{Q}$. Then the $\mathbf{Q}$-dynamics of $r$, $w$, $s$, and $u$ are given by

$$\begin{aligned}
dr(t) &= (\theta_r(t) + b_r w(t) - \hat{a}_r r(t))\, dt + \sigma_r d\widehat{W}_r(t), & (17)\\
dw(t) &= (\theta_w - \hat{a}_w w(t))\, dt + \sigma_w d\widehat{W}_w(t), & (18)\\
ds(t) &= (\theta_s + b_{su} u(t) - b_{sw} w(t) - \hat{a}_s s(t))\, dt + \sigma_s d\widehat{W}_s(t), & (19)\\
du(t) &= (\theta_u - \hat{a}_u u(t))\, dt + \sigma_u d\widehat{W}_u(t), & (20)
\end{aligned}$$

where $\hat{a}_i = a_i + \lambda_i \sigma_i^2$, $i = r, w, u, s$. Using Equation (1) and Equations (17) and (18) we can calculate the price of a non-defaultable zero-coupon bond in the extended Schmid and Zagst model:

**Theorem 3 (Price of a non-defaultable zero-coupon bond)** *The time $t$ price of a non-defaultable zero-coupon bond with maturity $T$ is given by*

$$P(t,T) = \mathbb{E}^{\mathbf{Q}}\left[ e^{-\int_t^T r(l)dl} \middle| \mathcal{F}_t \right] = P(t,T,r(t),w(t)),$$

*where*

$$P(t,T,r,w) = e^{A(t,T) - B(t,T)r - E(t,T)w}$$

*with*

$$\begin{aligned}
B(t,T) &= \frac{1}{\hat{a}_r}\left(1 - e^{-\hat{a}_r(T-t)}\right),\\
E(t,T) &= \frac{b_r}{\hat{a}_r}\left(\frac{1 - e^{-\hat{a}_w(T-t)}}{\hat{a}_w} + \frac{e^{-\hat{a}_w(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_w - \hat{a}_r}\right),\\
A(t,T) &= \int_t^T \frac{1}{2}\sigma_r^2 B(l,T)^2 + \frac{1}{2}\sigma_w^2 E(l,T)^2 - \theta_r(l) B(l,T) - \theta_w E(l,T)\, dl.
\end{aligned}$$

**Proof.**

Special case of the two-factor Hull-White model (see Hull & White (1994)). ■

Using Equation (2) and Equations (17) - (20) we can calculate the price of a defaultable zero-coupon bond in the extended Schmid and Zagst model:

**Theorem 4 (Price of a defaultable zero-coupon bond)** *The price of a defaultable zero-coupon bond at time $t < \min(T^d, T)$ is given by*

$$P^d(t,T) = \mathbb{E}^{\mathbf{Q}}\left[ e^{-\int_t^T (r(l)+s(l))dl} \middle| \mathcal{G}_t \right] = P^d(t,T,r(t),w(t),s(t),u(t)),$$

*where*

$$P^d(t,T,r,w,s,u) = e^{A^d(t,T) - B(t,T)r - E^d(t,T)w - C^d(t,T)s - D^d(t,T)u} \tag{21}$$

*with*

$$
\begin{aligned}
B(t,T) &= \frac{1}{\hat{a}_r}\left(1-e^{-\hat{a}_r(T-t)}\right), \\
C^d(t,T) &= \frac{1}{\hat{a}_s}\left(1-e^{-\hat{a}_s(T-t)}\right), \\
D^d(t,T) &= \frac{b_{su}}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_u(T-t)}}{\hat{a}_u}+\frac{e^{-\hat{a}_u(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_u-\hat{a}_s}\right), \\
E^d(t,T) &= \frac{b_r}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_w(T-t)}}{\hat{a}_w}+\frac{e^{-\hat{a}_w(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_w-\hat{a}_r}\right) \\
&\quad -\frac{b_{sw}}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_w(T-t)}}{\hat{a}_w}+\frac{e^{-\hat{a}_w(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_w-\hat{a}_s}\right), \\
A^d(t,T) &= \int_t^T \frac{1}{2}\left(\sigma_s^2 C^d(l,T)^2+\sigma_u^2 D^d(l,T)^2+\sigma_w^2 E^d(l,T)^2+\sigma_r^2 B(l,T)^2\right) \\
&\quad -\theta_r(l)B(l,T)-\theta_s C^d(l,T)-\theta_w E^d(l,T)-\theta_u D^d(l,T)dl.
\end{aligned}
$$

**Proof.**
See appendix. ■

Based on the formulas for defaultable and non-defaultable zero-coupon bonds we can easily calculate the yields to maturity $R(t,T)$ and $R^d(t,T)$ of non-defaultable and defaultable bonds as well as the yield spread $S(t,T)$ of defaultable bonds to non-defaultable bonds:

$$
\begin{aligned}
R(t,T) &= -\frac{A(t,T)}{T-t}+\frac{B(t,T)}{T-t}r+\frac{E(t,T)}{T-t}w, \\
R^d(t,T) &= -\frac{A^d(t,T)}{T-t}+\frac{B(t,T)}{T-t}r \\
&\quad +\frac{E^d(t,T)}{T-t}w+\frac{C^d(t,T)}{T-t}s+\frac{D^d(t,T)}{T-t}u,
\end{aligned}
$$

and

$$
\begin{aligned}
S(t,T) &= \frac{A(t,T)}{T-t}-\frac{A^d(t,T)}{T-t} \\
&\quad +\left(\frac{E^d(t,T)-E(t,T)}{T-t}\right)w \\
&\quad +\frac{C^d(t,T)}{T-t}s+\frac{D^d(t,T)}{T-t}u.
\end{aligned}
$$

$R(t,T)$ depends on the GDP growth rate $w$ with a positive weight $E(t,T)/(T-t)$. The model considers the empirically observable relationship between interest rates and the general condition of the economy in a reasonable way. $S(t,T)$ depends on the GDP growth rate $w$ with a negative weight

$(E^d(t,T) - E(t,T))/(T-t)$. Therefore, $w$ ensures the empirically validated negative correlation between spreads and non-defaultable short rates (see also Bakshi, Madan & Zhang (2001)).

Krishnan et al. (2005) suggest that credit spreads of different maturities for the same firm may move in different directions. In particular, the credit spread curve can move upward, downward, or reflect humped-shaped shocks. It is easy to show that the extended model of Schmid and Zagst can generate each of these specific term structures of credit spreads.

If we estimate the parameters using the data specified in Section 2 by application of Kalman filter techniques we get the estimates as specified in Tables 3 and 4.

Table 3: Results of the parameter estimations for the processes $r$ and $w$ in the extended model of Schmid and Zagst using non-defaultable weekly bond data from October 1, 1993, until June 1, 2001.

| | |
|---|---|
| $a_r$ | 0.07909401 |
| $b_r$ | 0.2078202 |
| $\sigma_r$ | 0.0133694 |
| $\theta_w$ | 0.022030853 |
| $a_w$ | 1.6291287 |
| $\sigma_w$ | 0.006265796 |
| $\hat{a}_r$ | 0.0772493132 |
| $\hat{a}_w$ | 0.4597083006 |

Table 4: Results of the parameter estimations for the processes $s$ and $u$ in the extended model of Schmid and Zagst using defaultable weekly bond data of American Industrials for the two rating classes A2, and BBB1 from October 1, 1993, until June 1, 2001.

| | BBB1 | A2 |
|---|---|---|
| $a_s$ | 2.574470 | 2.843421 |
| $\sigma_s$ | 0.0061187990 | 0.006284887 |
| $\theta_s$ | 0.0027527780 | 0.002396343 |
| $b_{sw}$ | 0.0986226500 | 0.0968989 |
| $a_u$ | 0.1293081000 | 0.138004 |
| $\sigma_u$ | 0.0071989270 | 0.007437778 |
| $\theta_u$ | 0.0020081870 | 0.002042065 |
| $\hat{a}_s$ | 1.067005 | 1.234556 |
| $\hat{a}_u$ | 0.04386892 | 0.02235216 |

Based on these parameter estimates we can calculate the mean reversion levels of the stochastic processes. For $w$ we get 1.35%, for $r$ we get an average mean reversion level of 5.17%. These values are quite intuitive as the average observed GDP growth rate is around 1.36% and the average observed 3−month rate is approximately 5.09%. For rating categories $BBB1$ and $A2$ the mean reversion levels of $u$ are 1.55%, 1.48%, and for $s$ 66 bp, 56 bp, respectively. These numbers are rather intuitive as the average 3−month spreads for rating categories $BBB1$ and $A2$ approximately equal 69 bp and 60 bp, respectively, i.e. decrease for ratings of higher quality.

# 5 The Model of Bakshi, Madan and Zhang

In the model of Bakshi et al. (2006) uncertainty is modeled with a three-dimensional standard Brownian motion $W = (W_r, W_w, W_u)'$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$. The dynamics of the non-defaultable short rate are described by a two-factor model

$$\begin{aligned} dr(t) &= (w(t) - a_r r(t))\,dt + \sigma_r \sqrt{1-\rho_{r,w}^2}\,dW_r(t) + \sigma_r \rho_{r,w}\,dW_w(t), \quad (22) \\ dw(t) &= (\theta_w - a_w w(t))\,dt + \sigma_w dW_w(t), \quad (23) \end{aligned}$$

where $a_r$, $a_w$, $\sigma_r$, and $\sigma_w$, are positive constants, $\theta_w \geq 0$ and $|\rho_{r,w}| < 1$. The mean reversion level of the short rate $r$ is dependent on $w$. $w$ is assumed to be unobservable. The short rate spread is modeled according to

$$ds(t) = (\Lambda_r - 1)dr(t) + \Lambda_u du(t), \quad (24)$$

where $u$ is given by

$$du(t) = (\theta_u - a_u u(t))\,dt + \sigma_u \frac{\rho_{r,u}}{\sqrt{1-\rho_{r,w}^2}} dW_r(t) + \sigma_u \sqrt{1 - \frac{\rho_{r,u}^2}{1-\rho_{r,w}^2}}\, dW_u(t). \quad (25)$$

$a_u$ and $\sigma_u$ are positive constants, $\theta_u \geq 0$, and $\rho_{r,u}^2 < 1 - \rho_{r,w}^2$. Therefore, the short rate spread is driven by a factor describing the general state of the economy and a firm specific component. Bakshi et al. (2001) use for $u$ firm-specific data like stock prices. Note that the system of stochastic differential equations as given by Equations (22) - (25) has a unique strong solution for each given initial value $(r_0, w_0, s_0, u_0)' \in \mathbb{R}^4$. If we now define a progressively measurable process $\gamma(t) = (\gamma_r(t), \gamma_w(t), \gamma_u(t))'$ such that

$$\begin{aligned} \gamma_w(t) &= \lambda_w \sigma_w w(t), \\ \gamma_r(t) &= \lambda_r \sigma_r r(t) - \frac{\rho_{r,w}}{\sqrt{1-\rho_{r,w}^2}} \gamma_w(t), \\ \gamma_u(t) &= \lambda_u \sigma_u u(t) - \frac{\frac{\rho_{r,u}}{\sqrt{1-\rho_{r,w}^2}}}{\sqrt{1 - \frac{\rho_{r,u}^2}{1-\rho_{r,w}^2}}} \gamma_r(t) = \lambda_u \sigma_u u(t) - \frac{\rho_{r,u}}{\sqrt{1-\rho_{r,w}^2-\rho_{r,u}^2}} \gamma_r(t) \end{aligned}$$

for real constants $\lambda_r$, $\lambda_w$, $\lambda_u$, by applying Girsanov's theorem we can show that

$$\widehat{W}(t) = W(t) + \int_0^t \gamma(l)dl$$

is a standard Brownian motion under the measure $\mathbf{Q}$. Then the $\mathbf{Q}$-dynamics of $r$, $w$, $s$, and $u$ are given by

$$\begin{aligned}
dr(t) &= (w(t) - \hat{a}_r r(t))\,dt + \sigma_r \sqrt{1-\rho_{r,w}^2}\,d\widehat{W}_r(t) + \sigma_r \rho_{r,w} d\widehat{W}_w(t),\\
dw(t) &= (\theta_w - \hat{a}_w w(t))\,dt + \sigma_w d\widehat{W}_w(t),\\
ds(t) &= (\Lambda_r - 1)dr(t) + \Lambda_u du(t),\\
du(t) &= (\theta_u - \hat{a}_u u(t))\,dt + \sigma_u \frac{\rho_{r,u}}{\sqrt{1-\rho_{r,w}^2}} d\widehat{W}_r(t) + \sigma_u \sqrt{1 - \frac{\rho_{r,u}^2}{1-\rho_{r,w}^2}}\,d\widehat{W}_u(t),
\end{aligned}$$

where $\hat{a}_r = a_r + \lambda_r \sigma_r^2 \sqrt{1-\rho_{r,w}^2}$, $\hat{a}_w = a_w + \lambda_w \sigma_w^2$, and $\hat{a}_u = a_u + \lambda_u \sigma_u^2 \sqrt{1 - \frac{\rho_{r,u}^2}{1-\rho_{r,w}^2}}$. Using Equation (1) and Equations (17) and (18) we can calculate the prices of non-defaultable and defaultable zero-coupon bonds in the model of Bakshi, Madan and Zhang:

**Theorem 5 (Price of a non-defaultable zero-coupon bond)** *The time $t$ price of a non-defaultable zero-coupon bond with maturity $T$ is given by*

$$P(t,T) = \mathbb{E}^{\mathbf{Q}}\left[ e^{-\int_t^T r(l)dl} \middle| \mathcal{F}_t \right] = P(t,T,r(t),w(t)),$$

*where*

$$P(t,T,r,w) = e^{A(t,T)-B(t,T)r(t)-E(t,T)w(t)}$$

*with*

$$\begin{aligned}
B(t,T) &= \frac{1}{\hat{a}_r}\left(1 - e^{-\hat{a}_r(T-t)}\right),\\
E(t,T) &= \frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_w(T-t)}}{\hat{a}_w} + \frac{e^{-\hat{a}_w(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_w - \hat{a}_r}\right),\\
A(t,T) &= \int_t^T \frac{1}{2}\sigma_r^2 B(l,T)^2 + \frac{1}{2}\sigma_w^2 E(l,T)^2 + \sigma_r \rho_{r,w} \sigma_w B(l,T)E(l,T)\\
&\quad -\theta_w E(l,T)\,dl.
\end{aligned}$$

**Proof.**

See Bakshi et al. (2001). ■

**Theorem 6 (Price of a defaultable zero-coupon bond)** *The price of a defaultable zero-coupon bond at time $t < \min(T^d, T)$ is given by*

$$P^d(t,T) = \mathbb{E}^{\mathbf{Q}}\left[ e^{-\int_t^T \Lambda_0 + \Lambda_r r(l) + \Lambda_u u(l)dl} \middle| \mathcal{F}_t \right] = P^d(t,T,r(t),w(t),u(t)),$$

*where*

$$P^d(t,T,r,w,u) = e^{A^d(t,T) - B^d(t,T)r(t) - E^d(t,T)w(t) - D^d(t,T)u(t)} \tag{26}$$

*with*

$$\begin{aligned}
B^d(t,T) &= \frac{\Lambda_r}{\hat{a}_r}\left(1 - e^{-\hat{a}_r(T-t)}\right),\\
D^d(t,T) &= \frac{\Lambda_u}{\hat{a}_u}\left(1 - e^{-\hat{a}_u(T-t)}\right),\\
E^d(t,T) &= \frac{\Lambda_r}{\hat{a}_r}\left(\frac{1 - e^{-\hat{a}_w(T-t)}}{\hat{a}_w} + \frac{e^{-\hat{a}_w(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_w - \hat{a}_r}\right),\\
A^d(t,T) &= \int_t^T \frac{1}{2}\left(\sigma_u^2 D^d(l,T)^2 + 2\sigma_u\rho_{r,u}\sigma_r B^d(l,T)D^d(l,T)\right.\\
&\quad \left. + \sigma_w^2 E^d(l,T)^2 + \sigma_r^2 B^d(l,T)^2 + 2\sigma_r\rho_{r,w}\sigma_w B^d(l,T)E^d(l,T)\right)\\
&\quad - \theta_u D^d(l,T) - \theta_w E^d(l,T) - \Lambda_0 dl.
\end{aligned}$$

**Proof.**
See appendix. ■

Similar to the extended model of Schmid and Zagst the non-defaultable short rate and the short-rate spread are negatively correlated (in all our parameter estimations $\Lambda_r$ is between 0 and 1).

Note that

$$\begin{aligned}
ds(t) &= (\Lambda_r - 1)dr(t) + \Lambda_u du(t)\\
&= \left(\Lambda_u\theta_u + (\Lambda_r - 1)\left(w(t) - a_r r(t)\right) - \Lambda_u a_u u(t)\right)dt + \sigma_s dZ_s(t)\\
&= \left(\theta_s - b_{sw}w(t) + b_{su}u(t) - a_s s(t)\right)dt + \sigma_s dZ_s(t),
\end{aligned}$$

where $\theta_s = \Lambda_0 a_r + \Lambda_u\theta_u$, $b_{su} = \Lambda_u\left(a_r - a_u\right)$, $b_{sw} = 1 - \Lambda_r$, $a_s = a_r$, and $Z_s$ is the Itô-process given by

$$\begin{aligned}
\sigma_s dZ_s(t) &= \left((\Lambda_r - 1)\sigma_r\sqrt{1-\rho_{r,w}^2} + \Lambda_u\sigma_u\frac{\rho_{ru}}{\sqrt{1-\rho_{r,w}^2}}\right)dW_r(t)\\
&\quad + (\Lambda_r - 1)\sigma_r\rho_{r,w}dW_w\\
&\quad + \Lambda_u\sigma_u\sqrt{1 - \frac{\rho_{ru}^2}{1-\rho_{r,w}^2}}dW_u(t).
\end{aligned}$$

Therefore, there is a close relationship between the stochastic differential equation for $s$ in the model of Bakshi, Madan and Zhang and the one in the extended model of Schmid and Zagst.

Table 5: Results of the parameter estimations for the processes $r$ and $w$ in the model of Bakshi, Madan and Zhang using non-defaultable weekly bond data from October 1, 1993, until June 1, 2001.

| | |
|---|---|
| $a_r$ | 0.250997971 |
| $\rho_{r,w}$ | -0.225356756 |
| $\sigma_r$ | 0.011050606 |
| $\theta_w$ | 0.003472108 |
| $a_w$ | 0.273347331 |
| $\sigma_w$ | 0.004563472 |
| $\hat{a}_r$ | 0.06400339 |
| $\hat{a}_w$ | 0.6426661 |

Table 6: Results of the parameter estimations for the processes $s$ and $u$ in the model of Bakshi, Madan and Zhang using defaultable weekly bond data of American Industrials for the two rating classes BBB1 and A2 from October 1, 1993, until June 1, 2001.

| | BBB1 | A2 |
|---|---|---|
| $a_u$ | 0.95207426 | 0.9796834 |
| $\sigma_u$ | 0.00188839 | 0.001516783 |
| $\theta_u$ | 0.002639271 | 0.002028697 |
| $\rho_{r,u}$ | 0.5346880518 | 0.5278927125 |
| $\Lambda_r$ | 0.961901623 | 0.939365879 |
| $\hat{a}_u$ | 0.25596 | 0.5397213 |

If we estimate the parameters using the data specified in Section 2, by application of Kalman filter techniques we get the estimates as specified in Tables 5 and 6.

Based on these parameter estimates we can calculate the mean reversion levels of the stochastic processes. For $w$ we get 1.27%, for $r$ we get an average mean reversion level of 5.06%. This value is quite intuitive as the average observed 3-month rate is approximately 5.09%. For rating categories $BBB1$ and $A2$ the mean reversion levels of $u$ are 28 bp, 21 bp, and for $s$ 62 bp, 58 bp, respectively. These numbers are rather intuitive as the average 3-month spreads for rating categories $BBB1$ and $A2$ approximately equal 69 bp and 60 bp, respectively, i.e. decrease for ratings of higher quality.

# 6 Comparison of the Model Performance

As a first empirical analysis we compare the average absolute deviations of the model prices of non-defaultable bond yields and credit spreads from their observed market prices. We use monthly data from October 1, 1993, to June 1, 2001, for an in-sample test and June 8, 2001, to December 31, 2004, for an out-of sample test. We consider non-defaultable bond yields and credit spreads of the following maturities: 1 year, 2, 3, 4, 5, 7, and 10 years.

- In the extended model of Schmid and Zagst $w$ is observable, whereas in the model of Bakshi, Madan and Zhang it is unobservable. Schmid and Zagst fit $w$ to the GDP growth rates and $r$ to non-defaultable interest rates (using observed values of $P$). Bakshi, Madan and Zhang estimate the parameters of $r$ as well as $w$ such that they match the market prices of the non-defaultable bonds giving them more degrees of freedom. Therefore, the average absolute deviation is slightly smaller than in the extended and in the original model of Schmid and Zagst. The results of the in-sample and out-of sample tests are given in Tables 7 and 8, respectively.

- In case of the spreads both models of Schmid and Zagst perform better than the model of Bakshi, Madan and Zhang for all rating classes. The additional factor $u$ helps to better match the observed data.

- We also regress observed changes in zero rates and spreads against the theoretical model prices in order to see how well observed changes in market prices can be explained by model prices. The $R^2$ values averaged over all maturities are given in Table 9 for the in-sample time period and in Table 10 for the out-of-sample time period.

Table 7: Average absolute deviations (in-sample, in bp) of the model and market prices of US Treasury Strips and spreads of different rating classes. We used data from October 1, 1993, to June 1, 2001. These absolute deviations compare to an average 3-month spread (insample) of 69 bp for the BBB1 and 60 bp for the A2 rating category, and 509 bp for the average 3-month Treasury rate (insample).

| Rating Category | SZ | Ext. SZ | Bakshi et al |
|---|---|---|---|
| Industrials BBB1 | 3.88 | 3.22 | 12.17 |
| Industrials A2 | 5.25 | 4.33 | 11.11 |
| Treasury Strips | 16.91 | 11.52 | 4.41 |

Table 8: Average absolute deviations (out-of-sample, in bp) of the model and market prices of US Treasury Strips and spreads of different rating classes. We used data from June 8, 2001, to December 31, 2004. These absolute deviations compare to an average 3-month spread (out-of-sample) of 92 bp for the BBB1 and 65 bp for the A2 rating category, and 214 bp for the average 3-month Treasury rate (out-of-sample).

| Rating Category | SZ | Ext. SZ | Bakshi et al |
|---|---|---|---|
| Industrials BBB1 | 4.01 | 2.68 | 8.13 |
| Industrials A2 | 3.92 | 3.95 | 11.58 |
| Treasury Strips | 54.86 | 68.18 | 10.56 |

Table 9: Average $R^2$values (in-sample) of the different models based on data from October 1, 1993, to June 1, 2001.

| Rating Class | SZ | Ext. SZ | Bakshi et al |
|---|---|---|---|
| Industrials BBB1 | 65.68 | 80.47 | 41.84 |
| Industrials A2 | 34.91 | 81.29 | 13.39 |
| Treasury Strips | 79.33 | 82.00 | 88.43 |

Table 10: Average $R^2$ values (out–of-sample) of the different models based on data from June 8, 2001, to December 31, 2004.

| Rating Class | SZ | Ext. SZ | Bakshi et al |
|---|---|---|---|
| Industrials BBB1 | 70.78 | 81.79 | 46.47 |
| Industrials A2 | 46.16 | 84.72 | 12.49 |
| Treasury Strips | 82.44 | 87.36 | 93.61 |

- The results of the regression tests for Treasury Strips support that the model of Bakshi, Madan and Zhang performs best for non-defaultable rates.
- Based on the regression tests for the spreads the models of Schmid and Zagst perform best not only in the sense of average values as given in Tables 9 and 10 but even for all maturities. This model is better capable of modeling the structural differences of bond prices through time. This is caused by the greater flexibility of the spread process. Our empirical analysis shows that the short-rate spread $s$ has a higher correlation with short-rate interest rates, whereas the uncertainty index $u$ has a higher correlation with long-term interest rates (see Table 12).

Table 11: Correlations of the processes $s$ and $u$ in the model of Schmid and Zagst with the observed BBB1 spreads for different maturities.

| | Observed spreads with maturities (in years) | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 7 | 10 |
| $s$ | 0.96 | 0.91 | 0.88 | 0.85 | 0.83 | 0.80 | 0.77 |
| $u$ | 0.96 | 0.99 | 1 | 1 | 1 | 0.99 | 0.98 |

Table 12: Correlations of the processes $s$ and $u$ in the extended model of Schmid and Zagst with the observed BBB1 spreads for different maturities.

| | Observed spreads with maturities (in years) | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 7 | 10 |
| $s$ | 0.90 | 0.84 | 0.79 | 0.76 | 0.73 | 0.69 | 0.65 |
| $u$ | 0.93 | 0.96 | 0.98 | 0.99 | 1.00 | 1.00 | 1.00 |

Table 13: Correlations of the process $u$ in the model of Bakshi, Madan and Zhang with the observed BBB1 spreads for different maturities.

| | Observed spreads with maturity (in years) | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 7 | 10 |
| $u$ | 0.99 | 0.99 | 0.99 | 0.98 | 0.97 | 0.96 | 0.95 |

- This result is further backed by the behavior of the functions $C^d(t,T)/(T-t)$ and $D^d(t,T)/(T-t)$ that basically show the dependence of $S(t,T)$ on $s$ and $u$ for fixed $t$. While both functions converge to 0 as $T$ tends to $\infty$, they show a different behavior in the time interval relevant for us (see Figures 6 and 7): $C^d(t,T)/(T-t)$ is monotonically decreasing towards 0, while $D^d(t,T)/(T-t)$ is hump-shaped. Therefore, short-maturity spreads highly depend on $s$, while long-term spreads highly depend on $u$. Such (from a structural perspective) interesting result cannot be found in the model of Bakshi, Madan und Zhang for both rating classes in such a distinct way, as in this model only one factor is fitted to the observed spreads (see Figure 8). In addition, this factor has a smaller correlation with the long-term spreads than the factor $u$ and the correlation structure changes only minimal over different maturities. (see Table 13)

Figures 9 and 10 show that the mean reversion levels of $s$ in the extended model of Schmid and Zagst and the process $u$ in the model of Bakshi, Madan and Zhang are good measures for the average default probability of the specific rating class. In addition, the processes $s$ can be seen as signaling or early warning processes for rating changes.

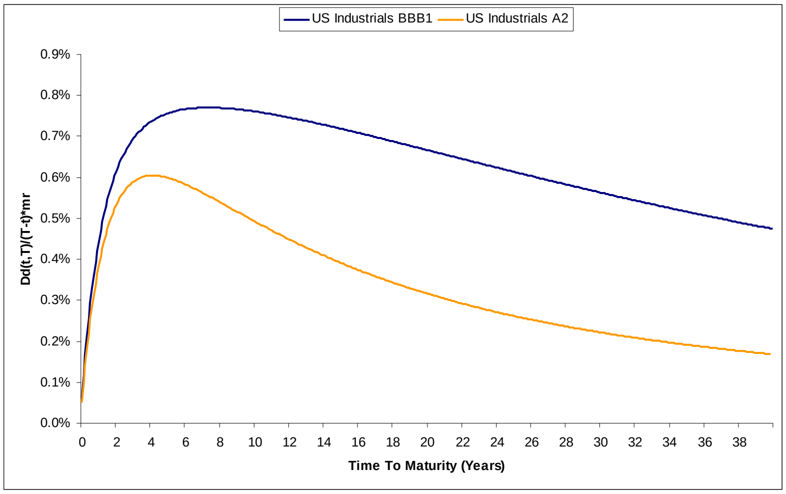


Figure 4: Plot of the function $\frac{D^d(t,T)}{T-t}$ from the model of Schmid and Zagst dependent on $T-t$ for rating class $BBB1$ and $A2$. This function shows how $R^d(t,T)$ depends on $u(t)$. The plot is standardized with the mean reversion levels of $u$.

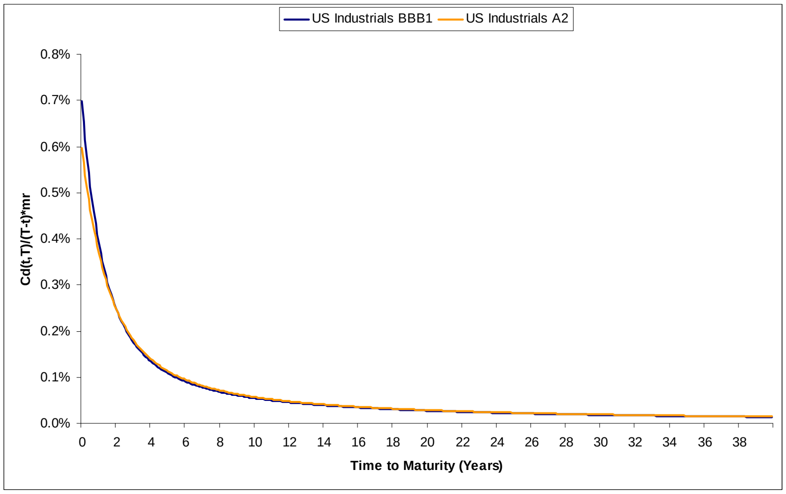


Figure 5: Plot of the function $\frac{C^d(t,T)}{T-t}$ from the model of Schmid and Zagst dependent on $T-t$ for rating class $BBB1$ and $A2$. This function shows how $R^d(t,T)$ depends on $s(t)$. The plot is standardized with the mean reversion levels of $s$.

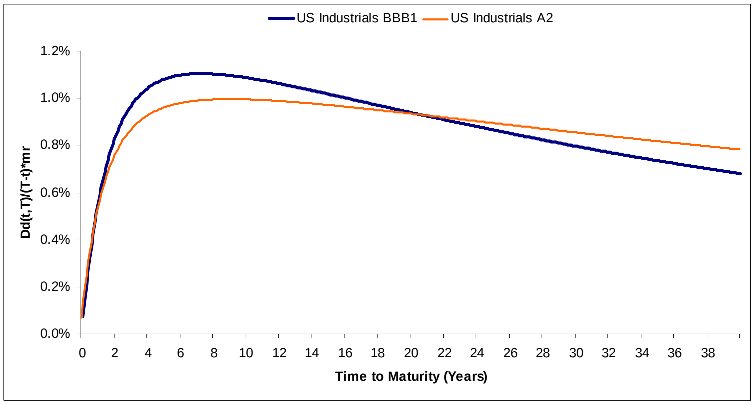


Figure 6: Plot of the function $\frac{D^d(t,T)}{T-t}$ from the extended model of Schmid and Zagst dependent on $T-t$ for rating class $BBB1$ and $A2$. This function shows how $R^d(t,T)$ depends on $u(t)$. The plot is standardized with the mean reversion levels of $u$.

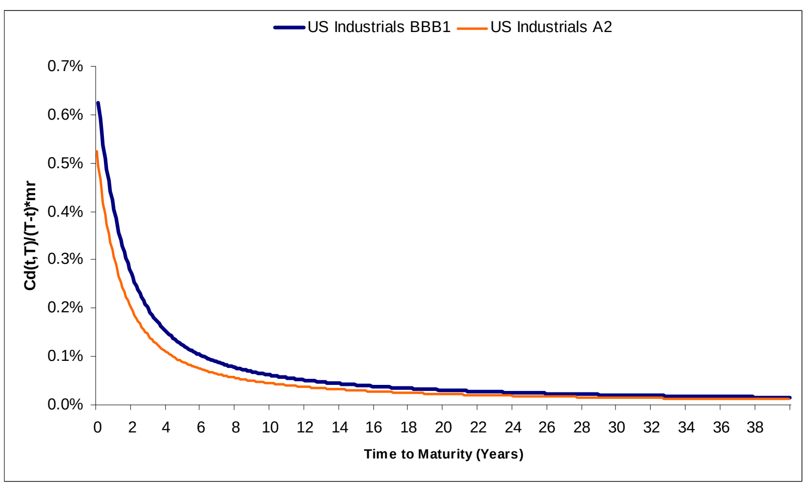


Figure 7: Plot of the function $\frac{C^d(t,T)}{T-t}$ from the extended model of Schmid and Zagst dependent on $T-t$ for rating classes $BBB1$ and $A2$. This function shows how $R^d(t,T)$ depends on $s(t)$. The plot is standardized with the mean reversion levels of $s$.

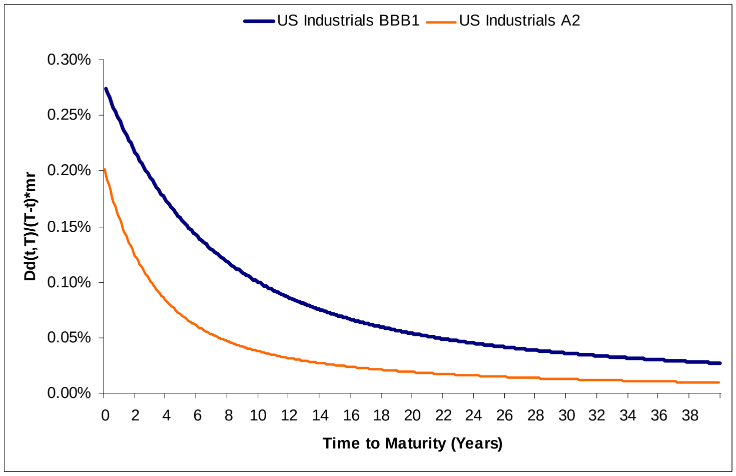


Figure 8: Plot of the function $\frac{D^d(t,T)}{T-t}$ from the model of Bakshi, Madan and Zhang dependent on $T-t$ for the rating categories BBB1 and A2. Basically, this function shows how $S(t,T)$ depends on $u(t)$. The plot is standardized with the mean reversion level of $u$.

# 7 Concluding Remarks

We have presented a new hybrid four-factor model and have compared its performance to other hybird models. The model is a straightforward extension of our three-factor model. We expicitly model firm-specific as well as systematic credit risk. With regards to the ability to explain observed credit spreads the four-factor model is superior to the three-factor model and the model of Bakshi et al. (2006). A firm-specific factor alone is not able to capture all changes in credit spreads. In fact, if credit spreads are modeled dependent on market and firm-specific variables, levels and changes of credit spreads can be explained much better.

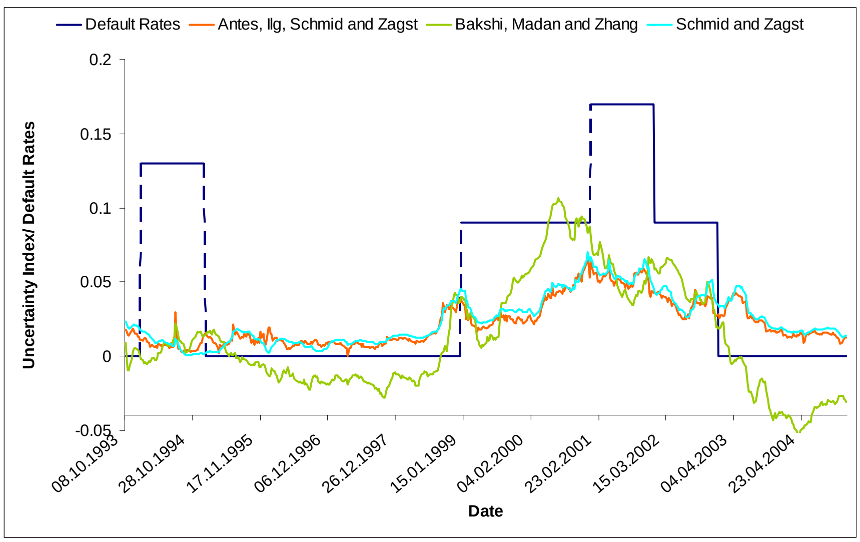


Figure 9: Historical average default probabilities of rating category $A$ (from Standard and Poor's), mean reversion level $(b_s u(t))/a_s$ of process $s$ in the model of Schmid and Zagst, mean reversion level $(\theta_s - b_{sw}w(t) + b_{su}u(t))/a_s$ of process $s$ in the extended model of Schmid and Zagst, and process $u$ in the model of Bakshi, Madan and Zhang, all based on our parameter estimates for $A2$ rated corporate bonds (data used for estimation: October 1, 1993 – June 1, 2001). The mean reversion levels of $s$ and the process $u$ are appropriately scaled. The horizontal axis is scaled in years, time 0 is October 1, 1993.

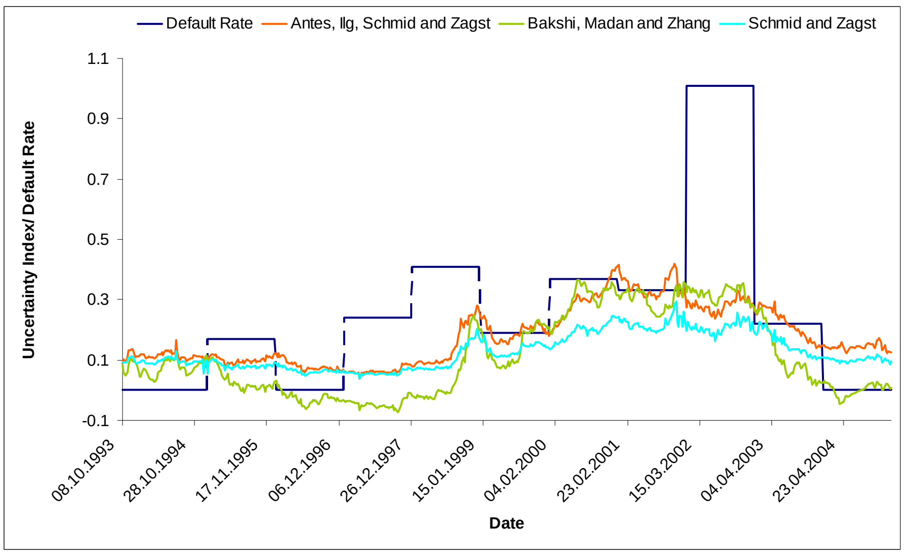


Figure 10: Historical average default probabilities of rating category $BBB$ (from Standard and Poor's), mean reversion level $(b_s u(t))/a_s$ of process $s$ in the model of Schmid and Zagst, mean reversion level $(\theta_s - b_{sw} w(t) + b_{su} u(t))/a_s$ of process $s$ in the extended model of Schmid and Zagst, and process $u$ in the model of Bakshi, Madan and Zhang, all based on our parameter estimates for $BBB1$ rated corporate bonds (data used for estimation: October 1, 1993 $-$ June 1, 2001). The mean reversion levels of $s$ and the process $u$ are appropriately scaled. The horizontal axis is scaled in years, time 0 is October 1, 1993.

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# A Definition of $v(t,T)$

$$
\begin{aligned}
v(t,T) &= \left(\sigma_u^2\right)^{2\phi(\kappa_1(s))} \frac{\varphi_1}{\varphi_2} \left(\sigma_u^2 e^{-\delta_s(T-t)}\right)^{\frac{\hat{a}_u}{2\delta_s} - \phi\left(\kappa_1^{(s)}\right)} F_1(t,T) \qquad (27) \\
&+ \left(\sigma_u^2 e^{-\delta_s(T-t)}\right)^{\frac{\hat{a}_u}{2\delta_s} + \phi\left(\kappa_1^{(s)}\right)} F_3(t,T),
\end{aligned}
$$

where

$$
\begin{aligned}
F_1(t,T) &= F\left(-\phi\left(\kappa_1^{(s)}\right) - \phi\left(\kappa_2^{(s)}\right), -\phi\left(\kappa_1^{(s)}\right) + \phi\left(\kappa_2^{(s)}\right),\right. \\
&\quad \left. 1 - 2\phi\left(\kappa_1^{(s)}\right), \kappa_2^{(s)}/\kappa_1^{(s)} e^{-\delta_s(T-t)}\right), \\
F_3(t,T) &= F\left(\phi\left(\kappa_1^{(s)}\right) - \phi\left(\kappa_2^{(s)}\right), \phi\left(\kappa_1^{(s)}\right) + \phi\left(\kappa_2^{(s)}\right),\right. \\
&\quad \left. 1 + 2\phi\left(\kappa_1^{(s)}\right), \kappa_2^{(s)}/\kappa_1^{(s)} e^{-\delta_s(T-t)}\right),
\end{aligned}
$$

with

$$
\phi(g) = \sqrt{\frac{\hat{a}_u^2 g + 2b_s\sigma_u^2}{4\delta_s^2 g}}
$$

and $F(a,b,c,z)$ is the hypergeometric function[3]. In addition,

$$
\begin{aligned}
\varphi_1(t,T) &= \zeta_2 e^{-\delta_s(T-t)} F_4(t,T) - \xi_1 F_3(t,T), \\
\varphi_2(t,T) &= \xi_2 F_1(t,T) - \zeta_1 e^{-\delta_s(T-t)} F_2(t,T),
\end{aligned}
$$

where

$$
\begin{aligned}
F_2(t,T) &= F\left(1 - \phi\left(\kappa_1^{(s)}\right) - \phi\left(\kappa_2^{(s)}\right), 1 - \phi\left(\kappa_1^{(s)}\right) + \phi\left(\kappa_2^{(s)}\right),\right. \\
&\quad \left. 2 - 2\phi\left(\kappa_1^{(s)}\right), \kappa_2^{(s)}/\kappa_1^{(s)} e^{-\delta_s(T-t)}\right), \\
F_4(t,T) &= F\left(1 + \phi\left(\kappa_1^{(s)}\right) - \phi\left(\kappa_2^{(s)}\right), 1 + \phi\left(\kappa_1^{(s)}\right) + \phi\left(\kappa_2^{(s)}\right),\right. \\
&\quad \left. 2 + 2\phi\left(\kappa_1^{(s)}\right), \kappa_2^{(s)}/\kappa_1^{(s)} e^{-\delta_s(T-t)}\right),
\end{aligned}
$$

[3] The hypergeometric function, usually denoted by $F$, has series expansion

$$
F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},
$$

where$(a)_0 = 1$, $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$, $n \in \mathbb{N}$, and is the solution of the hypergeometric differential equation

$$
z(1-z)y'' + [c - (a+b+1)z]y' - aby = 0.
$$

The hypergeometric function can be written as an integral

$$
F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt, \quad (c > b > 0),
$$

and is also known as the Gauss series or the Kummer series.

$$\xi_{1/2} = \left( \frac{\hat{a}_u}{2} \pm \delta_s \phi \left( \kappa_1^{(s)} \right) \right), \ \zeta_{1/2} = \delta_s \frac{\kappa_2^{(s)}}{\kappa_1^{(s)}} \frac{\phi^2 \left( \kappa_2^{(s)} \right) - \phi^2 \left( \kappa_1^{(s)} \right)}{1 \mp 2\phi \left( \kappa_1^{(s)} \right)},$$

and $\varphi_1 = \varphi_1 (T, T)$ and $\varphi_2 = \varphi_2 (T, T)$.

# B Proof of Theorem 4

By the theorem of Feynman Kac $P^d$ is the solution to the following partial differential equation:

$$\begin{aligned} 0 &= \frac{1}{2} \left( \sigma_s^2 P_{ss}^d + \sigma_u^2 P_{uu}^d + \sigma_w^2 P_{ww}^d + \sigma_r^2 P_{rr}^d \right) + \left( \theta_r(t) + b_r w - \hat{a}_r r \right) P_r^d \\ &\quad + \left( \theta_w - \hat{a}_w w \right) P_w^d + \left( \theta_u - \hat{a}_u u \right) P_u^d + \left( \theta_s + b_{su} u - b_{sw} w - \hat{a}_s s \right) P_s^d \\ &\quad - (r+s) P^d + P_t^d. \end{aligned}$$

If we assume that $P^d$ is of the form (21), then for $P^d > 0$:

$$\begin{aligned} 0 &= \frac{1}{2} \left( \sigma_s^2 C^{d2} + \sigma_u^2 D^{d2} + \sigma_w^2 E^{d2} + \sigma_r^2 B^2 \right) + r \left( \hat{a}_r B - 1 - B_t \right) \\ &\quad + w \left( \hat{a}_w E^d - b_r B + b_{sw} C^d - E_t^d \right) + u \left( \hat{a}_u D^d - b_{su} C^d - D_t^d \right) \\ &\quad + s \left( \hat{a}_s C^d - 1 - C_t^d \right) + A_t^d - \theta_r(t) B - \theta_s C^d - \theta_w E^d - \theta_u D^d. \end{aligned}$$

This PDE is equivalent to the following system of ODEs:

$$\begin{aligned} B_t &= \hat{a}_r B - 1 \\ C_t^d &= \hat{a}_s C^d - 1 \\ D_t^d &= \hat{a}_u D^d - b_{su} C^d \\ E_t^d &= \hat{a}_w E^d - b_r B + b_{sw} C^d \\ -A_t^d &= \frac{1}{2} \left( \sigma_s^2 C^{d2} + \sigma_u^2 D^{d2} + \sigma_w^2 E^{d2} + \sigma_r^2 B^2 \right) - \theta_r(t) B - \theta_s C^d - \theta_w E^d - \theta_u D^d. \end{aligned}$$

As $P^d(T,T) = 1$ for all $r, w, s, u \in \mathbb{R}$ we know $A^d(T,T) = B(T,T) = C^d(T,T) = D^d(T,t) = E^d(T,T) = 0$. Using the transformation $\tau = T - t$ gives the following solution:

$$
\begin{aligned}
B(t,T) &= e^{-\hat{a}_r(T-t)}\int_0^{T-t} e^{\hat{a}_r l}dl = e^{-\hat{a}_r(T-t)}\frac{1}{\hat{a}_r}\left(e^{\hat{a}_r(T-t)}-1\right)\\
&= \frac{1}{\hat{a}_r}\left(1-e^{-\hat{a}_r(T-t)}\right),\\
C^d(t,T) &= \frac{1}{\hat{a}_s}\left(1-e^{-\hat{a}_s(T-t)}\right),\\
D^d(t,T) &= e^{-\hat{a}_u(T-t)}\int_0^{T-t} e^{\hat{a}_u l} b_{su} C^d(0,l)dl\\
&= e^{-\hat{a}_u(T-t)}\int_0^{T-t} e^{\hat{a}_u l} b_{su}\frac{1}{\hat{a}_s}\left(1-e^{-\hat{a}_s(l)}\right)dl\\
&= e^{-\hat{a}_u(T-t)} b_{su}\frac{1}{\hat{a}_s}\left(\frac{1}{\hat{a}_u}\left(e^{\hat{a}_u(T-t)}-1\right)-\frac{1}{\hat{a}_u-\hat{a}_s}\left(e^{(\hat{a}_u-\hat{a}_s)(T-t)}-1\right)\right)\\
&= b_{su}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_u(T-t)}}{\hat{a}_u}+\frac{e^{-\hat{a}_u(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_u-\hat{a}_s}\right),\\
E^d(t,T) &= e^{-\hat{a}_w(T-t)}\int_0^{T-t} e^{\hat{a}_w l}\left(b_r B(0,l)-b_{sw}C^d(0,l)\right)dl\\
&= -b_{sw}\frac{1}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_w(T-t)}}{\hat{a}_w}+\frac{e^{-\hat{a}_w(T-t)}-e^{-\hat{a}_s(T-t)}}{\hat{a}_w-\hat{a}_s}\right)\\
&\quad +b_r\frac{1}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_w(T-t)}}{\hat{a}_w}+\frac{e^{-\hat{a}_w(T-t)}-e^{-\hat{a}_r(T-t)}}{\hat{a}_w-\hat{a}_r}\right),\\
A^d(t,T) &= \int_t^T \frac{1}{2}\left(\sigma_s^2 C^d(l,T)^2+\sigma_u^2 D^d(l,T)^2+\sigma_w^2 E^d(l,T)^2+\sigma_r^2 B(l,T)^2\right)\\
&\quad -\theta_r(l)B(l,T)-\theta_s C^d(l,T)-\theta_w E^d(l,T)-\theta_u D^d(l,T)dl.
\end{aligned}
$$

# C Proof of Theorem 6

By the theorem of Feynman Kac $P^d$ is the solution to the following partial differential equation:

$$
\begin{aligned}
0 &= \frac{1}{2}\left(\sigma_u^2 P^d_{uu}+2\sigma_u\rho_{r,u}\sigma_r P^d_{ru}+\sigma_w^2 P^d_{ww}+\sigma_r^2 P^d_{rr}+2\sigma_r\rho_{r,w}\sigma_w P^d_{rw}\right)+(w-\hat{a}_r r)P^d_r\\
&+ (\theta_w-\hat{a}_w w)P^d_w+(\theta_u-\hat{a}_u u)P^d_u-(\Lambda_0+\Lambda_r r+\Lambda_u u)P^d+P^d_t.
\end{aligned}
$$

If we assume that the structure of $P^d$ is of the type as in Equation (26), then for $P^d>0$ we get:

$$
\begin{aligned}
0 &= \frac{1}{2}\left(\sigma_u^2 D^{d2}+2\sigma_u\rho_{r,u}\sigma_r D^d B^d+\sigma_w^2 E^{d2}+\sigma_r^2 B^{d2}+2\sigma_r\rho_{r,w}\sigma_w E^d B^d\right)\\
&\quad +r\left(\hat{a}_r B^d-\Lambda_r-B^d_t\right)+w\left(\hat{a}_w E^d-B^d-E^d_t\right)+u\left(\hat{a}_u D^d-\Lambda_u-D^d_t\right)\\
&\quad +A^d_t-\theta_u D^d-\theta_w E^d-\Lambda_0.
\end{aligned}
$$

This PDE is equivalent to the following system of ODEs:

$$\begin{aligned}
B_t^d &= \hat{a}_r B^d - \Lambda_r \\
D_t^d &= \hat{a}_u D^d - \Lambda_u \\
E_t^d &= \hat{a}_w E^d - B^d \\
-A_t^d &= \frac{1}{2}\left(\sigma_u^2 D^{d2} + 2\sigma_u \rho_{r,u} \sigma_r D^d B^d + \sigma_w^2 E^{d2} + \sigma_r^2 B^{d2} + 2\sigma_r \rho_{r,w} \sigma_w E^d B^d\right) \\
&\quad -\theta_u D^d - \theta_w E^d - \Lambda_0.
\end{aligned}$$

As $P^d(T,T) = 1$ for all $r, w, u \in \mathbb{R}$ we get $A^d(T,T) = B^d(T,T) = D^d(T,T) = E^d(T,T) = 0$. Using the transformation $\tau = T - t$ we find

$$\begin{aligned}
B^d(t,T) &= e^{-\hat{a}_r(T-t)} \int_0^{T-t} e^{\hat{a}_r l} \Lambda_r dl = e^{-\hat{a}_r(T-t)} \frac{\Lambda_r}{\hat{a}_r} \left(e^{\hat{a}_r(T-t)} - 1\right) \\
&= \frac{\Lambda_r}{\hat{a}_r} \left(1 - e^{-\hat{a}_r(T-t)}\right), \\
D^d(t,T) &= \frac{\Lambda_u}{\hat{a}_u} \left(1 - e^{-\hat{a}_u(T-t)}\right), \\
E^d(t,T) &= e^{-\hat{a}_w(T-t)} \int_0^{T-t} e^{\hat{a}_w l} \left(B^d(0,l)\right) dl \\
&= \frac{\Lambda_r}{\hat{a}_r} \left(\frac{1 - e^{-\hat{a}_w(T-t)}}{\hat{a}_w} + \frac{e^{-\hat{a}_w(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_w - \hat{a}_r}\right), \\
A^d(t,T) &= \int_t^T \frac{1}{2} \left(\sigma_u^2 D^d(l,T)^2 + 2\sigma_u \rho_{r,u} \sigma_r B^d(l,T) D^d(l,T) + \sigma_w^2 E^d(l,T)^2\right. \\
&\quad \left. + \sigma_r^2 B^d(l,T)^2 + 2\sigma_r \rho_{r,w} \sigma_w B^d(l,T) E^d(l,T)\right) \\
&\quad -\theta_u D^d(l,T) - \theta_w E^d(l,T) - \Lambda_0 dl.
\end{aligned}$$