Pricing of Credit Derivatives

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Abstract

We show how to price credit default options and swaps based on a four-factor defaultable term-structure model. We derive the pricing functions and show how to calibrate the model to market prices. Basically, we need three pieces of information: the actual non-defaultable, the defaultable and the zero-recovery defaultable term structure. The first two pieces can be easily obtained from observable market data, the latter can be inferred from the other two. We illustrate the whole pricing process, from model specification and parameter estimation to the actual credit derivatives pricing.

Key words: defaultable term structure model, credit derivatives, parameter estimation

JEL classification: G13, E43

Introduction

In this paper, we develop pricing formulas for credit default options and swaps based on the extended Schmid and Zagst defaultable term structure model (see Schmid et al. (2006)) which is an extension of the model of Schmid and Zagst (2000). We review the underlying defaultable term structure model in Chapter 1. As a hybrid model it combines ideas of structural and reduced-form models which can actually be shown to coincide under certain conditions (see, e.g., Duffie and Lando (2001)). The model is mainly driven by a non-defaultable short rate and a short-rate credit spread. It is assumed that the level of the interest rates depends on a general market factor. One of the factors that determine the credit spread is the so-called uncertainty index which can be understood as an aggregation of all information on the quality of the

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firm currently available: The greater the value of the uncertainty process the lower the quality of the obligor. The uncertainty index models the idiosyncratic default risk of a counterparty. In addition, credit spreads are driven by the general market factor which can be interpreted as a measure for the systemic credit risk of a counterparty. By doing so we relate credit spreads to the business cycle. We assume that the spread between a defaultable and a non-defaultable bond is considerably driven by the uncertainty index and the general market factor but that there may be additional factors which influence the level of the spreads: at least the contractual provisions, liquidity and the premium demanded in the market for similar instruments have a great impact on credit spreads. Our approach seems to be reasonable in that credit spreads provide useful observable information on data upon which pricing models can be based. In addition, the model can be fitted directly to match the actual process followed by interest-rate credit spreads. The analytical solution obtained for defaultable bonds can be implemented easily in practice, as all the variables and parameters can be implied from market data.
In Chapter 2 we develop formulas for the pricing of credit default options and swaps. Therefore, we need the following data for all maturities $T > 0$: The default-free term structure of bond prices $P(t,T)$, the defaultable term structure of bond prices $P^d(t,T)$, and the defaultable term structure of bond prices under zero recovery $P^{d,zero}(t,T)$. The first piece of information, the default-free term structure, is easily obtained. Possible choices are government curves or swap curves in developed economies. The second piece is the defaultable term structure of the reference credit. Ideally it is obtained directly from the prices of the reference credit's bonds. Finally, the third piece of input data are the defaultable bond prices under zero recovery. These prices are usually unobservable. But we can derive the zero-recovery term structure of bond prices from the default-free and defaultable term structures of bond prices.
In Chapter 3 we will briefly review how to estimate the parameters of the underlying processes. Therefore, we use Kalman filtering techniques (see, e.g., Schmid et al. (2006), Oksendal (1998), pp. $79 - 106$, and Harvey (1989)). Finally, we fit our model to market prices of credit default swaps and calculate recovery rates implied by these market prices.

1 The Underlying Defaultable Term Structure Model

In the following, we assume that markets are frictionless and perfectly competitive, that trading takes place continuously, that there are no taxes, transaction costs, or informational asymmetries, and that investors act as price takers. To determine the prices of default options and swaps it is essential to use a defaultable term-structure model. Therefore, we fix a terminal time horizon T^*. Uncertainty in the financial market is modeled by a complete probability space $(\Omega, \mathcal{G}, \mathbf{Q})$ and all random variables and stochastic processes introduced below are defined on this probability space. We assume that $(\Omega, \mathcal{G}, \mathbf{Q})$ is equipped with three filtrations $\mathbb{H}$, $\mathbb{F}$, and $\mathbb{G}$, i.e. three increasing and right-continuous families of sub-σ-fields of $\mathcal{G}$. The default time T^d of an obligor is an arbitrary non-negative random variable on $(\Omega, \mathcal{G}, \mathbf{Q})$. For the sake of convenience we assume that $\mathbf{Q}\left(T^d = 0\right) = 0$ and $\mathbf{Q}\left(T^d > t\right) > 0$ for every $t \in (0, T^*]$. For a given default time T^d, we introduce the associated default indicator or hazard function $H(t) = \mathbf{1}_{\{T^d \leq t\}}$ and the survival indicator function $L(t) = 1 - H(t), t \in (0, T^*]$. Let $\mathbb{H} = (\mathcal{H}_t)_{0 \leq t \leq T^*}$ be the filtration generated by the process H. In addition, we define the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$ as the filtration generated by the multi-dimensional standard Brownian motion $W'(t) = (W_r(t), W_w(t), W_u(t), W_s(t))$ and $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T^*}$ as the enlarged filtration $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, i.e. for every t we set $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$. All filtrations are assumed to satisfy the usual conditions of completeness and right-continuity. For the sake of simplicity we furthermore assume that $\mathcal{F}_0$ is trivial. It should be emphasized that T^d is not necessarily a stopping time with respect to the filtration $\mathbb{F}$ but of course with respect to the filtration $\mathbb{G}$. If we assumed that T^d was a stopping time with $\mathbb{F}$, then it would be necessarily a predictable stopping time. This situation is the case in all traditional structural models.

We will assume throughout that for any $t \in (0, T^*]$ the $\sigma-$fields $\mathcal{F}_{T^*}$ and $\mathcal{H}_t$ are conditionally independent (under $\mathbf{Q}$) given $\mathcal{F}_t$. This is equivalent to the assumption that $\mathbb{F}$ has the so-called martingale invariance property with respect to $\mathbb{G}$, i.e. any $\mathbb{F}-$martingale follows also a $\mathbb{G}-$martingale (see Bielecki and Rutkowski (2004), p. 167). For the technical proofs we will use another condition which is also known to be equivalent to the martingale invariance property (see Bielecki and Rutkowski (2004), p. 242): For any $t \in (0, T^*]$ and any $\mathbf{Q}-$integrable $\mathcal{F}_{T^*}-$measurable random variable X we have $E^{\mathbf{Q}}[X|\ \mathcal{G}_t] = E^{\mathbf{Q}}[X|\ \mathcal{F}_t]$. The modeling takes already place after measure transformation, i.e. we assume that $\mathbf{Q}$ is a martingale measure and all discounted security price processes are $\mathbf{Q}-$martingales with respect to a suitable numéraire. As numéraire we choose the money-market account $B(t) = e^{\int_0^t r(l)dl}$, where $r(t)$ is the non-defaultable short rate. In the following, all processes are defined on the probability space $(\Omega, \mathcal{G}, \mathbf{Q})$.

Assumption 1 *The dynamics of the non-defaultable short rate are given by the following stochastic differential equation (SDE):*

$$dr(t) = [\theta_r(t) + b_{rw} w(t) - a_r r(t)]\, dt + \sigma_r dW_r(t), \ 0 \leq t \leq T^*, \tag{1}$$

where a_r, b_{rw}, $\sigma_r > 0$ are positive constants, and θ_r is a non–negative valued deterministic function.

Assumption 2 *The dynamics of the market factor are given by the following SDE:*

$$dw(t) = [\theta_w - a_w w(t)]\, dt + \sigma_w dW_w(t), \ \ 0 \leq t \leq T^*, \tag{2}$$

where a_w, $\sigma_w > 0$ are positive constants and θ_w is a non–negative constant.

Assumption 3 *The development of the uncertainty index is given by the following stochastic differential equation:*

$$du(t) = [\theta_u - a_u u(t)]\, dt + \sigma_u dW_u(t), \ \ 0 \leq t \leq T^*, \tag{3}$$

where a_u, $\sigma_u > 0$ are positive constants and θ_u is a non–negative constant.

Assumption 4 *The dynamics of the short-rate spread, i.e. the defaultable short rate minus the non-defaultable short rate, is given by the following stochastic differential equation:*

$$ds(t) = [\theta_s + b_{su} u(t) - b_{sw} w(t) - a_s s(t)]\, dt + \sigma_s dW_s(t), \ \ 0 \leq t \leq T^*, \tag{4}$$

where a_s, b_{su}, b_{sw}, $\sigma_s > 0$ are positive constants and θ_s is a non–negative constant.

Given Assumptions 1 and 2, the price of a non-defaultable zero-coupon bond is given by the following proposition.

Proposition 5 *Under Assumptions 1 and 2 the time t value of a non-defaultable zero-coupon bond with maturity T, $P(t,T) = P(r,w,t,T)$, is given by*

$$P(t,T) = e^{A(t,T) - B(t,T)r - E(t,T)w}$$

with

$$\begin{aligned}
B(t,T) &= \frac{1}{a_r} \cdot \left[1 - e^{-a_r(T-t)} \right], \\
E(t,T) &= \frac{b_{rw}}{a_r} \cdot \left(\frac{1 - e^{-a_w(T-t)}}{a_w} + \frac{e^{-a_w(T-t)} - e^{-a_r(T-t)}}{a_w - a_r} \right), \ \ \textit{and} \\
A(t,T) &= \int_t^T \frac{\sigma_r^2}{2} B^2(\tau,T) + \frac{\sigma_w^2}{2} E^2(\tau,T) - \theta_r(\tau) B(\tau,T) - \theta_w E(\tau,T)\, d\tau.
\end{aligned}$$

A proof of this statement can be found in Schmid et al. (2006). They also generalize the result of Proposition 5 to the pricing of defaultable zero-coupon bonds. Assuming a fractional recovery of market value[4], the following proposition holds.

[4] It is assumed that there is compensation in terms of equivalent defaultable bonds, which have not defaulted yet, i.e. the recovery rate is expressed as a fraction of the market value of the defaulted bond just prior to default. By equivalent we

Proposition 6 *Given the dynamics specified by equations (1)-(4), the value at time* $t < \tau = \min\left(T, T^d\right)$, $P^d(t,T) = P^d(r,w,s,u,t,T)$, *of a defaultable zero-coupon bond with maturity* T *is given by*

$$\begin{aligned} P^d(t,T) &= e^{A^d(t,T) - B(t,T)r - C(t,T)s - D(t,T)u - E^d(t,T)w} \\ &= P(t,T) \cdot e^{A^*(t,T) - C(t,T)s - D(t,T)u + E^*(t,T)w}, \end{aligned}$$

where $A(t,T)$, $B(t,T)$, *and* $E(t,T)$ *are given in Proposition 5,*

$$\begin{aligned} C(t,T) &= \frac{1}{a_s} \cdot \left[\, 1 - e^{-a_s(T-t)} \right], \\ D(t,T) &= \frac{b_{su}}{a_s} \cdot \left(\frac{1 - e^{-a_u(T-t)}}{a_u} + \frac{e^{-a_u(T-t)} - e^{-a_s(T-t)}}{a_u - a_s} \right), \\ E^d(t,T) &= E(t,T) - E^*(t,T), \\ E^*(t,T) &= \frac{b_{sw}}{a_s} \cdot \left(\frac{1 - e^{-a_w(T-t)}}{a_w} + \frac{e^{-a_w(T-t)} - e^{-a_s(T-t)}}{a_w - a_s} \right), \\ A^*(t,T) &= A^d(t,T) - A(t,T), \end{aligned}$$

and

$$\begin{aligned} A^d(t,T) = \int_t^T & \frac{1}{2}\sigma_s^2 C^2(\tau,T) + \frac{1}{2}\sigma_u^2 D^2(\tau,T) \\ & + \frac{1}{2}\sigma_r^2 B^2(\tau,T)^2 + \frac{1}{2}\sigma_w^2 E^d(\tau,T)^2 - \theta_s C(\tau,T) \\ & - \theta_u D(\tau,T) - \theta_r(\tau) B(\tau,T) - \theta_w E^d(\tau,T)\, d\tau. \end{aligned}$$

2 The Default Option Pricing Formulas

2.1 Bond Prices under Zero Recovery

Suppose we want to price an option on a defaultable zero-coupon bond, i.e. a so-called credit option. If the option is knocked out at default of the zero-coupon bond, the buyer of the option receives nothing. Hence, we can interpret the option as a defaultable investment with zero recovery. We show that we can determine the price of the option as the expected value of the promised cash flow at maturity of the option discounted at risky discount rates. As the

mean bonds with the same maturity, quality and face value. This model was mainly developed by Duffie and Singleton (1999) and applied, e.g., by Schönbucher (2000).

recovery rate of the option is different from the recovery rate of the reference defaultable zero-coupon bond the risky discount rates are not the same as in the case of the pricing of defaultable zero-coupon bonds. Hence, we have to find a short rate credit spread s^{zero} describing the credit spread process of an obligor which is equivalent to the issuer of the zero-coupon bond (especially of the same quality) but with zero recovery rate. Therefore, for pricing credit derivatives such as credit options we need the following data for all maturities $T > 0$:

- the default-free term structure of bond prices $P(t,T)$,
- the defaultable term structure of bond prices $P^d(t,T)$,
- the defaultable term structure of bond prices $P^{d,zero}(t,T)$ under zero recovery.

Assumption 7 *The zero-recovery short-rate spread s^{zero} is given by:*

$$(1 - z(t)) \cdot s^{zero}(t) = s(t), \ 0 \leq t \leq T^*, \tag{5}$$

where s is the short-rate spread process defined in equation (4) and $0 \leq z < 1$ is the recovery-rate process. Furthermore, we assume that s^{zero} is a sufficiently good approximation to the intensity of H.

It should be noted that this setup allows for negative interest rates and credit spreads. Hereby, we follow the argumentation of Schönbucher (2003) in that a Gaussian specification is still acceptible because of the analytical tractability that is gained. Also note that, under the assumption of positive credit spreads, it can be shown that s^{zero} is the intensity of H iff the zero-recovery defaultable money market account is a tradeable security (see, e.g. Schmid (2004), p. 230, for more details). Our setup should therefore be viewed as a local approximation to the real-world dynamics and in this sense, we will use s^{zero} as a substitute to the intensity of H in the sequel.

Proposition 8 *Let Y be a $\mathcal{F}_T-$measurable random variable with $E^{\mathbf{Q}}[|Y|^q] < \infty$ for some $q > 1$. Under the zero-recovery assumption, i.e. under the assumption that the contingent claim is knocked out at default of the reference credit asset, and with the stochastic processes specified for $r, w, s, u,$ and s^{zero}, the price process, $V_{L,T}$:*

$$V_{L,T}(t) = E^{\mathbf{Q}}\left[e^{-\int_t^T r(l)dl} \ Y \cdot L(T) \right| \mathcal{G}_t \Big], \ 0 \leq t < T,$$

is given by

$$V_{L,T}(t) = L(t) \cdot V_T(t),$$

where the adapted continuous process V_T is defined by

$$V_T(t) = E^{\mathbf{Q}}\left[e^{-\int_t^T (r(l)+s^{zero}(l))dl} Y \mid \mathcal{F}_t \right], \ 0 \leq t < T, \tag{6}$$

and $V_T(t) = 0$ for $t \geq T$. Hence, if there has been no default until time t, $V_{L,T}(t)$ must equal the expected value of riskless cash flows discounted at zero-recovery risky discount rates. Equation (6) has a unique solution in the space consisting of every semimartingale, J, such that $E^{\mathbf{Q}}\left[\sup_t |J_t|^q\right] < \infty$ for some $q > 1$.

PROOF. See Schmid (2004), p. 230, with $\mathcal{F}_t$ substituted by $\mathcal{G}_t$ for the result under the enlarged filtration $\mathbb{G}$ and apply the martingale invariance property to show equation (6). □

Suppose we want to price a contingent claim that promises to pay off Y at maturity time T of the contingent claim, if the reference credit asset hasn't defaulted until then, and zero in case of a default. Then, the time t price of the reference credit asset, given there has been no default so far, depends on the stochastic processes r, w, s, and u. But in addition, the price of the contingent claim depends on s^{zero}, because discounting is done with $e^{-\int_t^T (r(l)+s^{zero}(l))dl}$.

In the following we assume that $z(t)$ is a known constant, i.e. $z(t) = z$ for all $0 \leq t \leq T^*$. Then, the dynamics of the zero-recovery short-rate spread are given by

$$ds^{zero}(t) = \left[\theta_s^{zero} + b_{su}^{zero} u(t) - b_{sw}^{zero} w(t) - a_s s^{zero}(t)\right] dt + \sigma_s^{zero} dW_s(t),$$

where $\theta_s^{zero} = \frac{\theta_s}{1-z}$, $b_{su}^{zero} = \frac{b_{su}}{1-z}$, $b_{sw}^{zero} = \frac{b_{sw}}{1-z}$ and $\sigma_s^{zero} = \frac{\sigma_s}{1-z}$. Now we can calculate the zero-recovery zero-coupon bond prices

$$P^{d,zero}(t,T) = e^{A^{d,zero}(t,T) - B(t,T)r - C^{zero}(t,T)s^{zero} - D^{zero}(t,T)u - E^{d,zero}(t,T)w},$$

where $A^{d,zero}(t,T)$, $E^{d,zero}(t,T)$, $C^{zero}(t,T)$, and $D^{zero}(t,T)$ are given by the corresponding formulas for $A^d(t,T)$, $E^d(t,T)$, $C(t,T)$, and $D(t,T)$ with θ_s, b_{su}, b_{sw} and σ_s substituted by θ_s^{zero}, b_{su}^{zero}, b_{sw}^{zero} and σ_s^{zero}, respectively.

2.2 Default Put Options

A default (digital) put option is a credit derivative under which one party (the beneficiary) pays the other party (the guarantor) a fixed amount (lump-sum fee up-front). This is in exchange for the guarantor's promise to make a fixed or variable payment in the event of default in one or more reference assets to cover the full loss in default. As reference instruments for default put options we only consider defaultable zero-coupon bonds in this section, i.e. there is a payoff that is the difference between the face value and the market value (at default) of a reference credit asset (cash settlement). That is, the payoff at the time T^d of default is

$$Z\left(T^d\right) = 1 - P^d\left(T^d, T\right) = 1 - z \cdot P^d\left(T^d -, T\right).$$

In case of a default digital put option the payoff is equal to 1 in case of a credit event before or at maturity. For the pricing of this derivative, let us first assume that the payoff takes place at maturity of the contract. Using equation (6), it is straightforward to show that the time t price of the default digital put option is given by

$$V_T^{ddp}(t) = E^{\mathbf{Q}}\left[e^{-\int_t^T r(l)dl} H(T) \middle| \mathcal{G}_t \right] = P(t,T) - L(t) \cdot P^{d,zero}(t,T)$$

with $P^{d,zero}(t,T)$ denoting the bond price under zero recovery.

Theorem 9 *If the payoff takes place at default of the reference credit asset, the time t price of the default digital put option is given by*

$$E^{\mathbf{Q}}\left[\int_t^T e^{-\int_t^u r(l)dl} dH(u) \middle| \mathcal{G}_t \right] = L(t) \cdot V_{T^d}^{ddp}(t)$$

with

$$\begin{aligned} V_{T^d}^{ddp}(t) &= E^{\mathbf{Q}}\left[\int_t^T e^{-\int_t^u (r(l)+s^{zero}(l))dl} s^{zero}(u)\, du \middle| \mathcal{F}_t \right] \\ &= \int_t^T E^{\mathbf{Q}}\left[e^{-\int_t^u (r(l)+s^{zero}(l))dl} s^{zero}(u) \middle| \mathcal{F}_t \right] du. \end{aligned} \tag{7}$$

PROOF. See Schmid (2004), p. 243, with $\mathcal{F}_t$ substituted by $\mathcal{G}_t$ for the result under $\mathbb{G}$ and apply the martingale invariance property to conclude with equation (7). □

The following theorem shows how to calculate the expected value in (7).

Theorem 10

$$\begin{aligned} &v(r, s^{zero}, u, w, t, T) := E^{\mathbf{Q}}\left[e^{-\int_t^T (r(l)+s^{zero}(l))dl} s^{zero}(T) \middle| \mathcal{F}_t \right] \\ &= P^{d,zero}(t,T) \cdot (F(t,T) + H(t,T)s^{zero}(t) + I(t,T)u(t) + J(t,T)w(t)) \end{aligned}$$

with

$$
\begin{aligned}
H(t,T) &= e^{-a_s(T-t)}, \;\; J(t,T) = b_{sw}^{zero} \cdot \frac{e^{-a_s(T-t)} - e^{-a_w(T-t)}}{a_s - a_w}, \\
I(t,T) &= -b_{su}^{zero} \cdot \frac{e^{-a_s(T-t)} - e^{-a_u(T-t)}}{a_s - a_u}, \\
F(t,T) &= -\frac{1}{2}\left((\sigma_s^{zero} C(t,T))^2 + (\sigma_u D^{zero}(t,T))^2\right) \\
&\quad +\theta_w \cdot (E^{d,zero}(t,T) - E(t,T)) + \theta_s^{zero} C(t,T) \\
&\quad +\theta_u D^{zero}(t,T) - \int_t^T \sigma_w^2 E^{d,zero}(l,T) J(l,T) dl.
\end{aligned}
$$

PROOF. For ease of notation we omit the superscript ^{zero} in this proof. According to the theorem of Feynman-Kac (see, e.g., Duffie (1992), p. 241 – 244, or Zagst (2002), p. 38 – 41), v is the solution of the following PDE:

$$
\begin{aligned}
0 &= \frac{1}{2}\left(\sigma_s^2 v_{ss} + \sigma_u^2 v_{uu} + \sigma_w^2 v_{ww} + \sigma_r^2 v_{rr}\right) + (\theta_r(t) + b_r w - a_r r)\, v_r \\
&\quad + (\theta_w - a_w w)\, v_w + (\theta_u - a_u u)\, v_u + (\theta_s + b_{su} u - b_{sw} w - \hat{a}_s s)\, v_s \\
&\quad -(r+s)v + v_t
\end{aligned}
$$

under the condition $v(r,s,u,w,T,T) = s$. If

$$
\begin{aligned}
& v(r,s,u,w,t,T) \\
&= P^d(t,T)(F(t,T) + G(t,T)r + H(t,T)s + I(t,T)u + J(t,T)w) \\
&= e^{A^d(t,T) - B(t,T)r - E^d(t,T)w - C(t,T)s - D(t,T)u} \\
&\quad \cdot (F(t,T) + G(t,T)r + H(t,T)s + I(t,T)u + J(t,T)w),
\end{aligned}
$$

then

$$
\begin{aligned}
0 &= \frac{1}{2}\left(\sigma_s^2 C^2 + \sigma_u^2 D^2 + \sigma_w^2 \left(E^d\right)^2 + \sigma_r^2 B^2\right)(F + Gr + Hs + Iu + Jw) \\
&\quad +(-\sigma_s^2 CH - \sigma_u^2 DI - \sigma_w^2 E^d J - \sigma_r^2 BG) \\
&\quad + (\theta_r(t) + b_r w - \hat{a}_r r)\,(-B(F + Gr + Hs + Iu + Jw) + G)) \\
&\quad + (\theta_w - \hat{a}_w w)\left(-E^d(F + Gr + Hs + Iu + Jw) + J\right) \\
&\quad + (\theta_s + b_{su} u - b_{sw} w - \hat{a}_s s)\,(-C(F + Gr + Hs + Iu + Jw) + H) \\
&\quad + (\theta_u - \hat{a}_u u)\,(-D(F + Gr + Hs + Iu + Jw) + I) \\
&\quad -(r+s)(F + Gr + Hs + Iu + Jw) \\
&\quad +(F + Gr + Hs + Iu + Jw)(A_t^d - C_t s - D_t u - E_t^d w - B_t r) \\
&\quad +(F_t + G_t r + H_t s + I_t u + J_t w).
\end{aligned}
$$

This reduces to

$$
\begin{aligned}
0 = & -\sigma_s^2 CH - \sigma_u^2 DI - \sigma_w^2 E^d J - \sigma_r^2 BG + \\
& G\theta_r(t) + J\theta_w + H\theta_s + I\theta_u + F_t + r(-\hat{a}_r G + G_t) \\
& + w(-\hat{a}_w J + J_t + b_r G - b_{sw} H) + s(-\hat{a}_s H + H_t) \\
& + u(-\hat{a}_u I + I_t + b_{su} H)
\end{aligned}
$$

under the boundary conditions

$$
G(T,T) = 0,\ F(T,T) = 0,\ H(T,T) = 1,\ I(T,T) = 0,\ J(T,T) = 0.
$$

Therefore, we have to solve the following system of linear equations:

$$
\begin{aligned}
G_t = & a_r G,\ J_t = a_w J - b_r G + b_{sw} H,\ H_t = a_s H,\ I_t = a_u I - b_{su} H, \\
F_t = & \sigma_s^2 CH + \sigma_u^2 DI + \sigma_w^2 E^d J + \sigma_r^2 BG - G\theta_r(t) - J\theta_w - H\theta_s - I\theta_u.
\end{aligned}
$$

Applying the transformation $\tau = T - t$ we get:

$$
\begin{aligned}
G(t,T) = & 0,\ H(t,T) = e^{-\hat{a}_s(T-t)}, \\
J(t,T) = & e^{-\hat{a}_w(T-t)} \int_0^{T-t} -e^{\hat{a}_w l} b_{sw} e^{-\hat{a}_s l} dl = b_{sw} \cdot \frac{e^{-\hat{a}_s(T-t)} - e^{-\hat{a}_w(T-t)}}{\hat{a}_s - \hat{a}_w}, \\
I(t,T) = & e^{-\hat{a}_u(T-t)} \int_0^{T-t} e^{\hat{a}_u l} b_{su} e^{-\hat{a}_s l} dl = -b_{su} \cdot \frac{e^{-\hat{a}_s(T-t)} - e^{-\hat{a}_u(T-t)}}{\hat{a}_s - \hat{a}_u}, \\
F(t,T) = & \int_t^T -\sigma_s^2 C(l,T)H(l,T) - \sigma_u^2 D(l,T)I(l,T) \\
& -\sigma_w^2 E^d(l,T)J(l,T) - \sigma_r^2 B(l,T)G(l,T) \\
& +G(l,T)\theta_r(l) + J(l,T)\theta_w + H(l,T)\theta_s + I(l,T)\theta_u dl.
\end{aligned}
$$

Using

$$
-C_t = H, -D_t = I, -(E^d - E)_t = J
$$

we finally get

$$
\begin{aligned}
F(t,T) = & -\frac{1}{2}\left(\sigma_s^2 C(t,T)^2 + \sigma_u^2 D(t,T)^2\right) + \theta_w (E^d - E)(t,T) \\
& +\theta_s C(t,T) + \theta_u D(t,T) - \int_t^T \sigma_w^2 E^d(l,T)J(l,T)dl.
\end{aligned}
$$

□

Finally, we consider the case of a default put option on a zero-coupon bond which replaces the difference to par if default is triggered.

Theorem 11 *We assume that the underlying reference asset is a zero-coupon bond with maturity T^* and that the default put has maturity T. Then, for $t < T \leq T^*$ and replacement to the difference of par, we find the time t price of the default put to be*

$$V^{dp}(t) = L(t) \cdot \left(V_{T^d}^{ddp}(t) - P^d(t, T^*) + P^{d,*}(t, T, T^*) \right),$$

where

$$P^{d,*}(t, T, T^*) = e^{A^{d,*}(t,T,T^*) - B(t,T^*)r - C^*(t,T,T^*)s - D^*(t,T,T^*)u - E^{d,*}(t,T,T^*)w}$$

with

$$C^*(t, T, T^*) = \frac{1}{1-z} \cdot \left(C(t, T^*) - z e^{-a_s(T-t)} C(T, T^*) \right), \tag{8}$$

$$\begin{aligned} D^*(t, T, T^*) = {} & e^{-a_u(T-t)} D(T, T^*) + \frac{1}{1-z} D(t, T) \\ & - b_{su} C(T, T^*) \left[\frac{e^{-a_s(T-t)} - e^{-a_u(T-t)}}{a_s - a_u} \right], \end{aligned} \tag{9}$$

$$\begin{aligned} E^{d,*}(t, T, T^*) = {} & E(t, T^*) - e^{-a_w(T-t)} E^*(T, T^*) - \frac{1}{1-z} E^*(t, T) \\ & + b_{sw} C(T, T^*) \left(\frac{e^{-a_w(T-t)} - e^{-a_s(T-t)}}{a_w - a_s} \right), \end{aligned} \tag{10}$$

and

$$\begin{aligned} A^{d,*}(t, T, T^*) = {} & A^d(T, T^*) + \int_t^T \frac{\sigma_r^2}{2} B^2(\tau, T^*) + \frac{\sigma_s^2}{2} \left(C^*(\tau, T, T^*) \right)^2 d\tau \\ & + \int_t^T \frac{\sigma_u^2}{2} \left(D^*(\tau, T, T^*) \right)^2 + \frac{\sigma_w^2}{2} \left(E^{d,*}(\tau, T, T^*) \right)^2 \\ & - \int_t^T \theta_r(\tau) B(\tau, T^*) d\tau + \theta_s C^*(\tau, T, T^*) d\tau \\ & - \int_t^T \theta_u D^*(\tau, T, T^*) + \theta_w E^{d,*}(\tau, T, T^*) d\tau. \end{aligned}$$

PROOF. Let Z describe the payoff of the underlying bond upon default. Then, we get:

$$
\begin{aligned}
V^{dp}(t) =& E^Q\left[\int_t^T e^{-\int_t^u r(l)dl}\left(1-Z(u)\right)dH(u)\bigg|\mathcal{G}_t\right]\\
=& L(t)\cdot V^{ddp}_{T^d}(t) - E^Q\left[\int_t^T e^{-\int_t^u r(l)dl} Z(u)\,dH(u)\bigg|\mathcal{G}_t\right]\\
=& L(t)\cdot V^{ddp}_{T^d}(t) - E^Q\left[\int_t^{T^*} e^{-\int_t^u r(l)dl} Z(u)\,dH(u)\bigg|\mathcal{G}_t\right]\\
&+E^Q\left[e^{-\int_t^T r(l)dl}\int_T^{T^*} e^{-\int_T^u r(l)dl} Z(u)\,dH(u)\bigg|\mathcal{G}_t\right]\\
=& L(t)\cdot\left(V^{ddp}_{T^d}(t) - P^d(t,T^*) + P^{d,zero}(t,T^*)\right)\\
&+E^Q\left[e^{-\int_t^T r(l)dl}E^Q\left[\int_T^{T^*} e^{-\int_T^u r(l)dl} Z(u)\,dH(u)\bigg|\mathcal{G}_T\right]\bigg|\mathcal{G}_t\right]\\
&+E^Q\left[e^{-\int_t^T r(l)dl}E^Q\left[e^{-\int_T^{T^*} r(l)dl} L(T^*)\bigg|\mathcal{G}_T\right]\bigg|\mathcal{G}_t\right]\\
&-E^Q\left[e^{-\int_t^T r(l)dl}E^Q\left[e^{-\int_T^{T^*} r(l)dl} L(T^*)\bigg|\mathcal{G}_T\right]\bigg|\mathcal{G}_t\right]\\
=& L(t)\cdot\left(V^{ddp}_{T^d}(t) - P^d(t,T^*) + P^{d,zero}(t,T^*)\right)\\
&+E^Q\left[e^{-\int_t^T r(l)dl}P^d(T,T^*)\cdot L(T)\bigg|\mathcal{G}_t\right]\\
&-E^Q\left[e^{-\int_t^T r(l)dl}P^{d,zero}(T,T^*)\cdot L(T)\bigg|\mathcal{G}_t\right].
\end{aligned}
$$

By applying Proposition 8 we get

$$
\begin{aligned}
V^{dp}(t) =& L(t)\cdot\left(V^{ddp}_{T^d}(t) - P^d(t,T^*) + P^{d,zero}(t,T^*)\right)\\
&+L(t)\cdot E^Q\left[e^{-\int_t^T (r(l)+s^{zero}(l))dl}P^d(T,T^*)\bigg|\mathcal{F}_t\right]\\
&-L(t)\cdot E^Q\left[e^{-\int_t^T (r(l)+s^{zero}(l))dl}P^{d,zero}(T,T^*)\bigg|\mathcal{F}_t\right]
\end{aligned}
$$

which is equivalent to

$$V^{dp}(t) = L(t)\cdot\left(V^{ddp}_{T^d}(t) - P^d(t,T^*) + P^{d,*}(t,T,T^*)\right),$$

where

$$P^{d,*}(t,T,T^*) := E^Q\left[e^{-\int_t^T (r(l)+s^{zero}(l))dl}P^d(T,T^*)\bigg|\mathcal{F}_t\right].$$

To calculate $P^{d,*}$ we assume that there exists a solution of the type

$$P^{d,*}(t,T,T^*) = e^{A^{d,*}(t,T,T^*)-B^*(t,T,T^*)r-C^*(t,T,T^*)s-D^*(t,T,T^*)u-E^{d,*}(t,T,T^*)w}$$

and apply the theorem of Feynman-Kac to derive a system of correspong linear equations:

$$
\begin{aligned}
a_r B^*(t,T,T^*) - B_t^*(t,T,T^*) - 1 &= 0 \\
a_s C^*(t,T,T^*) - C_t^*(t,T,T^*) - \frac{1}{1-z} &= 0 \\
a_u D^*(t,T,T^*) - b_{su} C^*(t,T,T^*) - D_t^*(t,T,T^*) &= 0 \\
a_w E^{d,*}(t,T,T^*) - b_{rw} B^*(t,T,T^*) + b_{sw} C^*(t,T,T^*) - E_t^{d,*}(t,T,T^*) &= 0
\end{aligned}
$$

and

$$
\begin{aligned}
-A_t^{d,*}(t,T,T^*) = &\frac{1}{2}\Big(\sigma_r^2\left(B^*(t,T,T^*)\right)^2 + \sigma_s^2\left(C^*(t,T,T^*)\right)^2 \\
&+\sigma_u^2\left(D^*(t,T,T^*)\right)^2 + \sigma_w^2\left(E^{d,*}(t,T,T^*)\right)^2\Big) \\
&-\theta_r(t)\, B^*(t,T,T^*) - \theta_s C^*(t,T,T^*) \\
&-\theta_u D^*(t,T,T^*) - \theta_w E^{d,*}(t,T,T^*),
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
A^{d,*}(T,T,T^*) &= A^d(T,T^*),\ \ B^*(T,T,T^*) = B(T,T^*), \\
C^*(T,T,T^*) &= C(T,T^*),\ \ D^*(T,T,T^*) = D(T,T^*), \\
E^{d,*}(T,T,T^*) &= E^d(T,T^*).
\end{aligned}
$$

- Solving for B^* :

$$
B^*(t,T,T^*) = e^{-a_r(T-t)}\left(\frac{1}{a_r}\left(1-e^{-a_r(T^*-T)}\right) + \int_0^{T-t} e^{a_r l}\,dl\right) = B(t,T^*).
$$

- Solving for C^* : Let

$$
\widehat{C}(t,T,T^*) = C^*(t,T,T^*)\cdot(1-z)\,.
$$

Then

$$
a_s C^*(t,T,T^*) - C_t^*(t,T,T^*) - \frac{1}{1-z} = 0
$$

is equivalent to

$$
a_s \widehat{C}(t,T,T^*) - \widehat{C}_t(t,T,T^*) - 1 = 0.
$$

Using the theorem of Feynman-Kac, we can easily see from Proposition 6 that

$$
C(t,T^*) = a_s C_t(t,T^*) - 1.
$$

Hence, using Equation (8),

$$
\begin{aligned}
a_s \widehat{C}(t,T,T^*) - \widehat{C}_t(t,T,T^*) = &\, a_s C(t,T^*) - a_s z e^{-a_s(T-t)} C(T,T^*) \\
&-C_t(t,T^*) + a_s z e^{-a_s(T-t)} C(T,T^*) \\
= &\, a_s C(t,T^*) - C_t(t,T^*) = 1
\end{aligned}
$$

In addition,

$$\widehat{C}(T,T,T^*) = C(T,T^*)\,(1-z) = C^*(T,T,T^*)\,(1-z)$$

which is equivalent to

$$C^*(T,T,T^*) = C(T,T^*).$$

- Solving for D^* : Using Equation (9)

$$\begin{aligned}
&a_u D^*(t,T,T^*) - D_t^*(t,T,T^*) \\
=& a_u e^{-a_u(T-t)} D(T,T^*) + \frac{a_u}{1-z} D(t,T) \\
&-a_u b_{su} C(T,T^*) \left[\frac{e^{-a_s(T-t)} - e^{-a_u(T-t)}}{a_s - a_u} \right] \\
&-a_u e^{-a_u(T-t)} D(T,T^*) - \frac{1}{1-z} D_t(t,T) \\
&+b_{su} C(T,T^*) \left[\frac{a_s e^{-a_s(T-t)} - a_u e^{-a_u(T-t)}}{a_s - a_u} \right] \\
=& \frac{1}{1-z} \left(a_u D(t,T) - D_t(t,T) \right) + b_{su} C(T,T^*) e^{-a_s(T-t)}.
\end{aligned}$$

Again, using the theorem of Feynman-Kac, it can be easily seen from Proposition 6 that

$$D_t(t,T) = a_u D(t,T) - b_{su} C(t,T).$$

Hence,

$$\begin{aligned}
&a_u D^*(t,T,T^*) - D_t^*(t,T,T^*) \\
=& \frac{b_{su}}{1-z} \left(C(t,T) + (1-z)\, C(T,T^*) e^{-a_s(T-t)} \right) \\
=& \frac{b_{su}}{1-z} \left(C(t,T^*) - z C(T,T^*) e^{-a_s(T-t)} \right) \\
=& b_{su} C^*(t,T,T^*).
\end{aligned}$$

In addition,

$$D^*(T,T,T^*) = D(T,T^*) + \frac{1}{1-z} D\,(T,T) = D(T,T^*).$$

- Solving for $E^{d,*}$: Using Equation (10)

$$\begin{aligned}
&a_w E^{d,*}(t,T,T^*) - E_t^{d,*}(t,T,T^*)\\
=\,&a_w E(t,T^*) - a_w e^{-a_w(T-t)} E^*\left(T,T^*\right) - \frac{a_w}{1-z} E^*\left(t,T\right)\\
&+b_{sw} C\left(T,T^*\right)\left(\frac{a_w e^{-a_w(T-t)} - a_w e^{-a_s(T-t)}}{a_w - a_s}\right)\\
&-E_t(t,T^*) + a_w e^{-a_w(T-t)} E^*\left(T,T^*\right) + \frac{1}{1-z} E_t^*\left(t,T\right)\\
&-b_{sw} C\left(T,T^*\right)\left(\frac{a_w e^{-a_w(T-t)} - a_s e^{-a_s(T-t)}}{a_w - a_s}\right)\\
=\,&a_w E(t,T^*) - E_t(t,T^*) - b_{sw} C\left(T,T^*\right) e^{-a_s(T-t)}\\
&+\frac{1}{1-z}\Big(a_w E^d\left(t,T\right) - E_t^d\left(t,T\right) - \left(a_w E\left(t,T\right) - E_t\left(t,T\right)\right)\Big).
\end{aligned}$$

Using the theorem of Feynman-Kac, we know from Propositons 5 and 6 that

$$\begin{aligned}
E_t\left(t,T\right) &= a_w E\left(t,T\right) - b_{rw} B\left(t,T\right) \text{ and}\\
E_t^d\left(t,T\right) &= a_w E^d\left(t,T\right) - b_{rw} B\left(t,T\right) + b_{sw} C\left(t,T\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
a_w E^{d,*}(t,T,T^*) - E_t^{d,*}(t,T,T^*) &= b_{rw} B\left(t,T^*\right) - \frac{b_{sw}}{1-z} C\left(t,T\right)\\
&\quad -b_{sw} C\left(T,T^*\right) e^{-a_s(T-t)}\\
&= b_{rw} B\left(t,T^*\right) - \frac{b_{sw}}{1-z} C\left(t,T\right)\\
&\quad -b_{sw} C\left(T,T^*\right) e^{-a_s(T-t)}\\
&= b_{rw} B\left(t,T^*\right) - \frac{b_{sw}}{1-z} C\left(t,T^*\right)\\
&\quad -\frac{b_{sw} z}{1-z} C\left(T,T^*\right) e^{-a_s(T-t)}
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
a_w E^{d,*}(t,T,T^*) - E_t^{d,*}(t,T,T^*) &= b_{rw} B\left(t,T^*\right) - b_{sw} C^*\left(t,T,T^*\right)\\
&= b_{rw} B^*\left(t,T,T^*\right) - b_{sw} C^*\left(t,T,T^*\right).
\end{aligned}$$

In addition,

$$\begin{aligned}
E^{d,*}(T,T,T^*) &= E\left(T,T^*\right) - E^*\left(T,T^*\right) - \frac{1}{1-z} E^*\left(T,T\right)\\
&= E\left(T,T^*\right) - E^*\left(T,T^*\right) = E^d\left(T,T^*\right).
\end{aligned}$$

- Solving for $A^{d,*}$:

$$
\begin{aligned}
A^{d,*}(t,T,T^*) - A^{d,*}(T,T,T^*) &= A^{d,*}(t,T,T^*) - A^{d}\left(T,T\right) \\
&= -\int_t^T A_t\left(\tau,T,T^*\right) d\tau
\end{aligned}
$$

and the solution for $A^{d,*}$ follows by simple integration. $\square$

Note, that $P^{d,*}\left(t,T,T\right) = P^{d,zero}\left(t,T\right)$. Also note, that the result of the previous theorem can be easily generalized to the case of coupon-paying bonds.

2.3 Default Swaps

A default swap is a swap under which one party (the beneficiary) pays the other party (the guarantor) regular fees, amounts that are based on a generic interest rate, called the default swap spread or the default swap rate. This is in exchange for the guarantor's promise to make a fixed or variable payment in the event of default in one or more reference assets to cover the full loss in default. As reference instruments we only consider defaultable (zero-coupon) bonds. In practice, default swap contracts differ in their specific default payments. We assume replacement of the difference to par which is currently market standard. A default swap on a defaultable coupon bond therefore pays off the difference between par and the post-default coupon bond price. There is only principal but no coupon protection. The pricing of a default swap consists of two problems. At origination there is no exchange of cash flows and we have to determine the default swap spread S that makes the market value of the default swap zero. After origination, the market value of the default swap will change due to changes in the underlying variables. So, given the default swap spread S, we have to determine the current market value of the default swap. We assume throughout that the credit swap counterparties (beneficiary and guarantor) are default-free.

2.3.1 The underlying reference credit asset is a defaultable zero-coupon bond

We assume that the underlying reference credit asset is a defaultable zero-coupon bond with maturity T^* and that there has been no credit event until time t. In case of a credit default swap with maturity $T \leq T^*$ there are regular payments S (the credit swap spread) instead of an up-front fee $V^{dp}\left(0\right)$. The value of paying $V^{dp}\left(0\right)$ at the origination of the credit-default put option must be the same as paying S at some predefined times $t \leq t_1 \leq \cdots \leq t_m = T$ until a default happens. Note, that t_i, $i = 1, ..., m$, are all possible payment dates.

Hence,

$$V^{dp}(0) = S \cdot \sum_{i=1}^{m} P^{d,zero}(0, t_i).$$

This is equivalent to a credit swap spread of

$$S = \frac{V^{dp}(0)}{\sum_{i=1}^{m} P^{d,zero}(0, t_i)}.$$

2.3.2 The underlying reference credit asset is a defaultable coupon bond

In the following we assume that the underlying reference asset is a defaultable coupon bond. In addition, we assume that there has been no credit event until time t.

(1) Default put options and replacement of the difference to par:
The reference credit asset is a coupon bond with maturity T^* and discrete coupon payments c_i occurring at dates $t \leq \tau_1 \leq \cdots \leq \tau_n = T^*$. Then, the pricing argument for the default put option with maturity $T \leq T^*$ is exactly the same as in the case of the zero-coupon bond, and we get

$$V_c^{dp}(t) = V_{T^d}^{ddp}(t) - P_c^d(t, T^*) + P_c^{d,*}(t, T, T^*),$$

where

$$P_c^d(t, T^*) = \sum_{i=1}^{n} c_i P^d(t, \tau_i) + P^d(t, T^*)$$

and

$$P_c^{d,*}(t, T, T^*) = \sum_{i=1}^{n} c_i P^{d,*}(t, T, \tau_i) + P^{d,*}(t, T, T^*)$$

with $P^{d,*}(t, T, \tau_i) := P^{d,zero}(t, \tau_i)$ if $\tau_i \leq T$, $i = 1, \ldots, n$.

(2) Default swaps:
The credit swap spread can be calculated by the same argument as in the case of zero-coupon bonds. Hence, if there are regular payments S_c at some predefined times $t \leq t_1 \leq \cdots \leq t_m = T$ until a default happens, S_c is given by

$$S_c = \frac{V_c^{dp}(0)}{\sum_{i=1}^{m} P^{d,zero}(0, t_i)}. \tag{11}$$

3 Data and Parameter Estimation

3.1 Estimation of the Parameters of the Underlying Processes

We estimate the parameters of the underlying processes from observable time series of defaultable and non-defaultable zero rates (derived from quoted par yields). As our main data source we use Bloomberg. All prices are in US dollars so that we do not have to deal with currency risks at all. For the non-defaultable zero rates we use US Treasury Strips. For defaultable zero rates we consider average rates of American Industrials of the three rating classes AA, $A2$ and $BBB1$. We use weekly data from October 1, 1993, until December 31, 2004. All parameters are estimated using Kalman filter methodologies as suggested, e.g., by Schmid (2004). In addition to the interest-rate data, we consider quarterly growth rates of the US GDP. As there is no weekly data observable, we generate the missing data using a linear interpolation with a three-quarter time lag consistent to the market standard for the use of the consumer price index in the pricing of inflation-linked bonds. For all our estimations we use the software package S-PLUS finmetrics. The results of the parameter estimates are summarized in Tables 1 - 4. As the process u is unobservable, we set $b_{su} = 1$. More details with respect to parameter estimations as well as in- and out-of-sample tests of the model can be found in Schmid et al. (2006).

$\mathbf{a}_r$	$\boldsymbol{\sigma}_r$	$\mathbf{b}_r$
$0,24782465$	$0,01495526$	$0,1331524$

Table 1: Parameter estimates of the process r.

$\mathbf{a}_w$	$\boldsymbol{\sigma}_w$	$\boldsymbol{\theta}_w$
$0,268466581$	$0,006014616$	$0,015833145$

Table 2: Parameter estimates of the process w.

Rating	$\mathbf{a}_u$	$\boldsymbol{\sigma}_u$	$\boldsymbol{\theta}_u$
AA	$0,0687767$	$0,004591962$	$0,001217304$
A2	$0,0634092$	$0,004760403$	$0,001856633$
BBB1	$0,0666961$	$0,00488623$	$0,001897959$

Table 3: Parameter estimates of the process u.

Rating	$\mathbf{a}_s$	$\boldsymbol{\sigma}_s$	$\boldsymbol{\theta}_s$	$\mathbf{b}_{sw}$
AA	$0,5969643$	$0,003885464$	$0,002747381$	$0,073732082$
A2	$0,4661833$	$0,003087475$	$0,002513915$	$0,076344989$
BBB1	$0,391333$	$0,002867852$	$0,00237781$	$0,093691615$

Table 4: Parameter estimates of the process s.

3.2 Calibrating the Model to Market Prices of Credit Default Swaps

Finally we want to calibrate our model to observable market prices of liquid credit default swaps. We consider the following credit derivatives and underlying bonds:

- Credit Default Swap: Kimberly Clark, 5 years maturity; Underlying bond: Rating AA, Coupon 6,875% (semi-annual), Maturity: 15.02.2014, Issue date: 18.02.94, 1st coupon date: 15.08.94
- Credit Default Swap: Caterpillar, 5 years maturity; Underlying bond: Rating A2, Coupon 6,55% (semi-annual), Maturity: 01.05.2011, Issue date: 08.05.2001, 1st coupon date: 01.11.01
- Credit Default Swap: Masco Corporation, 5 years maturity; Underlying bond: Rating BBB1, Coupon 5,875% (semi-annual), Maturity: 15.07.2012, Issue date: 24.06.02, 1st coupon date: 15.01.2003

The bid and ask spreads of the credit default swaps for thirteen points in time can be seen in Figures 1 - 3. Based on the historical parameter estimations of the underlying processes we estimate the recovery rate parameter z such that the quadratic deviations of the market prices from the model prices are minimal. The estimates are 87% for Kimberly Clark, 89% for Caterpillar, and 90% for Masco Corporation. Note, that these estimates are for recovery rates expressed as percentages of market values of bonds prior to default. These estimates correspond to recovery values expressed as a percentage of par of around 40%. Based on the recovery-rate estimates, the model prices of the credit-default swap spreads implied by the recovery parameter estimates are visualized in Figures 1 - 3. The deviations of the model from the market spreads goes back to the fact that we used average historical data for the estimation of the parameters of the underlying processes. Therefore, in order to match the market prices we adjust the volatility parameter of the uncertainty index at each point in time. Figures 4 - 6 show the price deviations to the ask prices before volatility adjustment as well as the corresponding adjusted values for σ_u for each point in time and each of the three counterparties. Note that volatility σ_u influences both prices, $V_c^{dp}(0)$ and $\sum_{i=1}^{m} P^{d,zero}(0,t_i)$, in (11) leading to a non-monotonic dependence of S_c from σ_u.

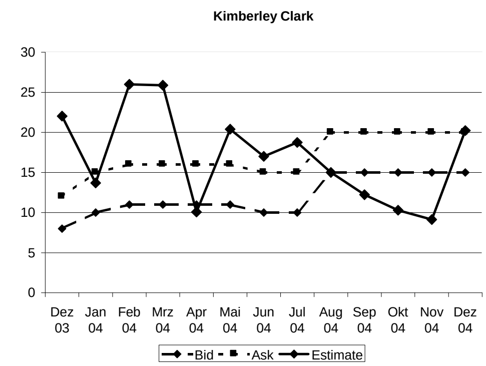

Fig. 1. Market and model credit default spreads in basis points (bp) for Kimberley Clark.

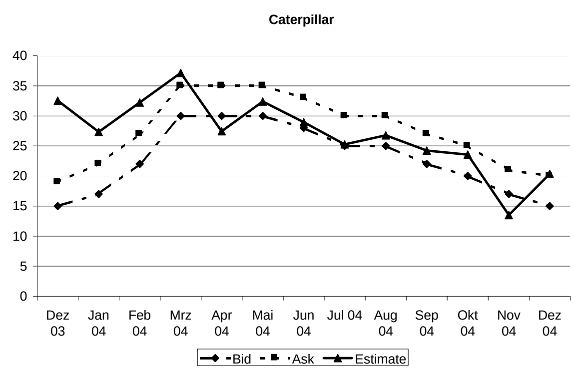

Fig. 2. Market and model credit default spreads (in bp) for Caterpillar.

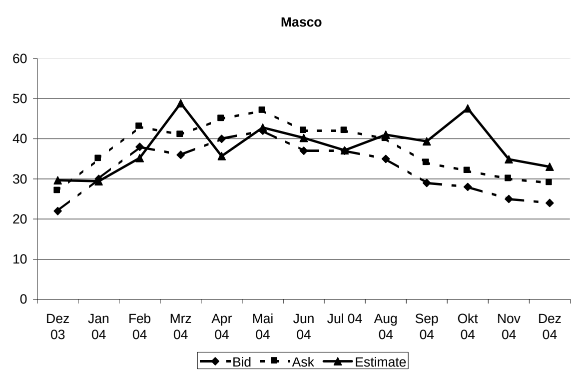

Fig. 3. Market and model credit default spreads (in bp) for Masco Corporation.

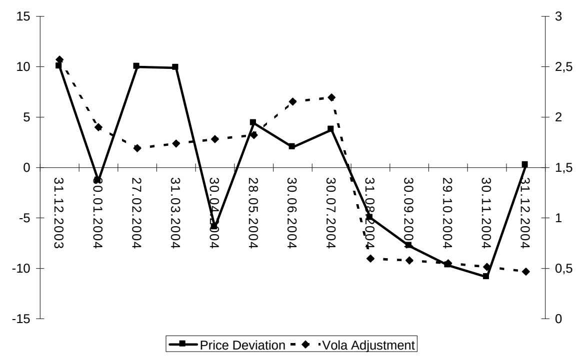

Fig. 4. Price deviation (in bp, left axis) and volatility adjustment (in bp, right axis) for Kimberly Clark.

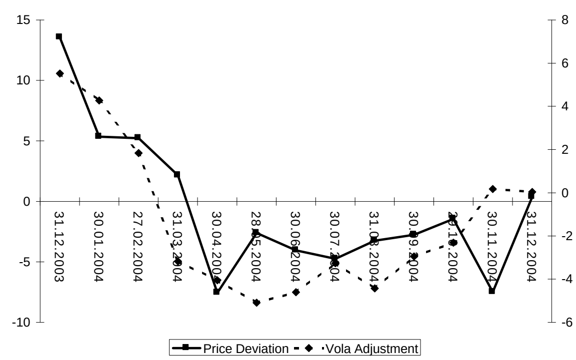

Fig. 5. Price deviation (in bp, left axis) and volatility adjustment (in bp, right axis) for Caterpillar.

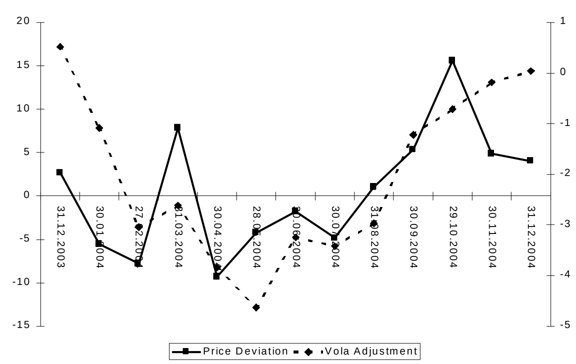

Fig. 6. Price deviation (in bp, left axis) and volatility adjustment (in bp, right axis) for Masco Corporation.

4 Summary and Conclusion

We developped pricing formulas for credit default options and swaps based on the extended Schmid and Zagst defaultable term-structure model. Hereby, we related credit spreads to the business cycle and assumed that the spread between a defaultable and a non-defaultable bond is considerably driven by an uncertainty index modeling the idiosyncratic default risk of a counterparty. We fitted our model to market prices of credit default swaps and calculated recovery rates implied by these market prices. As we apply average historical data for estimating the parameters of the underlying processes, we used the volatility of the uncertainty index to adjust the model to market prices. According to the small adjustments of the volatility parameter over time, the model seems to do a very good job in explaining the market prices of CDS.

References

Bielecki, T.R. and M. Rutkowski, 2004, Credit risk: modeling, valuation and hedging, Springer Finance (Springer, Berlin et al.).

Duffie, D., 1992, Dynamic asset pricing theory (Princeton University Press, Princeton).

Duffie, D. and D. Lando, 2001, Term structures of credit spreads with incomplete accounting information, Econometrica 69 (3), 633-664.

Duffie, D. and K.J. Singleton, 1999, Modeling term structures of defaultable bonds, Review of Financial Studies 12, 687-720.

Harvey, A.C., 1989, Forecasting, structural time series models and the Kalman filter (Cambridge University Press, Cambridge).

Hull, J. and A. White, 1990, Pricing interest-rate-derivative securities, The Review of Financial Studies 3 (4), 573-592.

Nelson, C. and A. Siegel, 1987, Parsimonious modeling of yield curves, Journal of Business 60, 473-489.

Oksendal, B., 1998, Stochastic differential equations - an introduction with applications (Springer, Berlin et al.).

Schmid, B., 2004, Credit risk pricing models, Springer Finance (Springer, Berlin et al.).

Schmid, B. and R. Zagst, 2000, A three-factor defaultable term structure model, The Journal of Fixed Income 10 (2), 63-79.

Schmid, B., Zagst, R., Antes, S. and M. Ilg, 2006, Empirical evaluation of hybrid defaultable bond pricing models, Working Paper.

Schönbucher, P.J., 2000, Credit risk modelling and credit derivatives, PhD thesis, Rheinische Friedrich-Wilhelms-Universität Bonn.

Schönbucher, P.J., 2003, Credit derivatives pricing models: models, pricing, and implementation, Wiley Finance Series (John Wiley & Sons, Chichester et al.).

Zagst, R., 20002, Interest rate management, Springer Finance (Springer, Berlin et al.).