Modeling and Pricing of Credit Derivatives Using Macro-Economic Information

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Abstract

We show how to price credit default options and swaps based on a four-factor defaultable term-structure model. One of the key factors is a macro-economic factor that takes into account the impact of the general economy on the quality of firms. We derive the pricing functions and show how to calibrate the model to market prices. Basically, we need three pieces of information: the actual non-defaultable, the defaultable and the zero-recovery defaultable term structure. The first two pieces can be easily obtained from observable market data, the latter can be inferred from the other two. We illustrate the whole pricing process, from model specification and parameter estimation to the actual credit derivatives pricing. Our data includes the recent credit crisis and proves the performance of our model even through times of market dislocation.

Key words: defaultable term structure model, credit derivatives, parameter estimation

JEL classification: G13, E43

Introduction

In this paper, we develop pricing formulas for credit default options and swaps based on the extended Schmid and Zagst defaultable term structure model (see Schmid et al. (2008)) which is an extension of the model of Schmid and Zagst (2000). We review the underlying defaultable term structure model in Chapter 1. As a hybrid model it combines ideas of structural and reduced-form models which can actually be shown to coincide under certain conditions (see, e.g., Duffie and Lando (2001)). The model is mainly driven by a non-defaultable short rate and a short-rate credit spread. It is assumed that the level of the interest rates depends on a general market factor. One of the factors that determine the credit spread is the so-called uncertainty index which can be understood as an aggregation of all information on the quality of the firm currently available: The greater the value of the uncertainty process the lower the quality of the obligor. The uncertainty index models the idiosyncratic default risk of a counterparty. In addition, credit spreads are driven by the general market factor which can be interpreted as a measure for the systemic credit risk of a counterparty. By doing

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so we relate credit spreads to the business cycle. We assume that the spread between a defaultable and a non-defaultable bond is considerably driven by the uncertainty index and the general market factor but that there may be additional factors which influence the level of the spreads: at least the contractual provisions, liquidity and the premium demanded in the market for similar instruments have a great impact on credit spreads. Our approach seems to be reasonable in that credit spreads provide useful observable information on data upon which pricing models can be based. In addition, the model can be fitted directly to match the actual process followed by interest-rate credit spreads. The analytical solution obtained for defaultable bonds can be implemented easily in practice, as all the variables and parameters can be implied from market data.

In Chapter 2 we develop formulas for the pricing of credit default options and swaps. Therefore, we need the following data for all maturities $T > 0$: The default-free term structure of bond prices $P(t,T)$, the defaultable term structure of bond prices $P^d(t,T)$, and the defaultable term structure of bond prices under zero recovery $P^{d,zero}(t,T)$. The first piece of information, the default-free term structure, is easily obtained. Possible choices are government curves or swap curves in developed economies. The second piece is the defaultable term structure of the reference credit. Ideally it is obtained directly from the prices of the reference credit's bonds. Finally, the third piece of input data is the defaultable bond prices under zero recovery. These prices are usually unobservable. But we can derive the zero-recovery term structure of bond prices from the default-free and defaultable term structures of bond prices.

In Chapter 3 we will briefly review how to estimate the parameters of the underlying processes based on historical data between 2002 and 2008. Therefore, we use Kalman filtering techniques (see, e.g., Schmid et al. (2008), Oksendal (1998), and Harvey (1989)). Furthermore, we fit our model to market prices of credit default swaps during the credit crisis of 2007/2008. We show how our model picks up latest market signals and performs well even through times of very volatile markets.

1 The Underlying Defaultable Term Structure Model

In the following, we assume that markets are frictionless and perfectly competitive, that trading takes place continuously, that there are no taxes, transaction costs, or informational asymmetries, and that investors act as price takers. To determine the prices of default options and swaps it is essential to use a defaultable term-structure model. Therefore, we fix a terminal time horizon T^*. Uncertainty in the financial market is modeled by a complete probability space $(\Omega,\mathbb{G},\mathbb{Q})$ and all random variables and stochastic processes introduced below are defined on this probability space. We assume that $(\Omega,\mathbb{G},\mathbb{Q})$ is equipped with three filtrations $\mathbb{H}$, $\mathbb{F}$, and $\mathbb{G}$, i.e. three increasing and right-continuous families of sub-σ-fields of $\mathbb{G}$. The default time T^d of an obligor is an arbitrary non-negative random variable on $(\Omega,\mathbb{G},\mathbb{Q})$. For the sake of convenience we assume that $\mathbb{Q}(T^d = 0) = 0$ and $\mathbb{Q}(T^d > t) > 0$ for every $t\in(0,T^*]$. For a given default time T^d we introduce the associated default indicator or hazard function $H(t) = 1_{\{T^d < t\}}$ and the survival indicator function $L(t) = 1 - H(t),\ t \in (0,T^*]$. Let $\mathbb{H} = (\mathcal{H}_t)_{0\le t\le T^*}$ be the filtration generated by the process H. In addition, we define the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T^*}$ as the filtration generated by the multi-dimensional standard Brownian motion

W'=(W_r,W_ω,W_u,W_s) and $\mathbb{G}=(\mathcal{G}_t)_{0\le t\le T^*}$ as the enlarged filtration $\mathbb{G}=\mathbb{H}\vee\mathbb{F}$, i.e. for every t we set $\mathcal{G}_t=\mathcal{H}_t\vee\mathcal{F}_t$. All filtrations are assumed to satisfy the usual conditions of completeness and right-continuity. For the sake of simplicity we furthermore assume that $\mathcal{F}_0$ is trivial. It should be emphasized that T^d is not necessarily a stopping time with respect to the filtration $\mathbb{F}$ but of course with respect to the filtration $\mathbb{G}$. If we assumed that T^d was a stopping time with $\mathbb{F}$, then it would be necessarily a predictable stopping time. This situation is the case in all traditional structural models.

We will assume throughout that for any $t\in(0,T^*]$ the σ-fields $\mathcal{F}_{T^*}$ and $\mathcal{H}_t$ are conditionally independent (under $\mathbb{Q}$) given $\mathcal{F}_t$. This is equivalent to the assumption that $\mathbb{F}$ has the so-called martingale invariance property with respect to $\mathbb{G}$, i.e. any $\mathbb{F}$-martingale follows also a $\mathbb{G}$-martingale (see Bielecki and Rutkowski (2004), p. 167). For the technical proofs we will use another condition which is also known to be equivalent to the martingale invariance property (see Bielecki and Rutkowski (2004), p. 242): For any $t\in(0,T^*]$ and any $\mathbb{Q}$-integrable $\mathcal{F}_{T^*}$-measurable random variable X we have $\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}_t]=\mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}_t]$. The modeling takes already place after measure transformation, i.e. we assume that $\mathbb{Q}$ is a martingale measure and all discounted security price processes are $\mathbb{Q}$-martingales with respect to a suitable numéraire. As numéraire we choose the money-market account $B(t)=\exp(\int_0^t r(l)dl)$, where r is the non-defaultable short rate. In the following, all processes are defined on the probability space $(\Omega,\mathbb{G},\mathbb{Q})$.

Assumption 1 The dynamics of the non-defaultable short rate are given by the following stochastic differential equation (SDE):

$$dr(t)=[\theta_r(t)+b_{r\omega}\omega(t)-a_r r(t)]dt+\sigma_r dW_r(t),\ \ 0\le t\le T^* \qquad (1)$$

where a_r, $b_{r\omega}$, σ_r are positive constants, and θ_r is a non-negative valued deterministic function.

Assumption 2 The dynamics of the market factor are given by the following SDE:

$$d\omega(t)=[\theta_\omega-a_\omega\omega(t)]dt+\sigma_\omega dW_\omega(t),\ \ 0\le t\le T^* \qquad (2)$$

where a_ω, σ_ω are positive constants and θ_ω is a non-negative constant.

Assumption 3 The development of the uncertainty index is given by the following stochastic differential equation:

$$du(t)=[\theta_u-a_u u(t)]dt+\sigma_u dW_u(t),\ \ 0\le t\le T^* \qquad (3)$$

where a_u, σ_u are positive constants and θ_u is a non-negative constant.

Assumption 4 The dynamics of the short-rate spread, i.e. the defaultable short rate minus the non-defaultable short rate, is given by the following stochastic differential equation:

$$ds(t) = [\theta_s + b_{su}u(t) - b_{s\omega}\omega(t) - a_s s(t)]dt + \sigma_s dW_s(t), \ \ 0 \le t \le T^* \qquad (4)$$

where a_s, b_{su}, $b_{s\omega}$, σ_s are positive constants, and θ_s is a non-negative constant.

Given Assumptions 1 and 2, the price of a non-defaultable zero-coupon bond is given by the following proposition.

Proposition 5 Under Assumptions 1 and 2 the time t value of a non-defaultable zero-coupon bond with maturity T, P(t,T)=P(r,ω,t,T) is given by

$$P(t,T) = e^{A(t,T)-B(t,T)r(t)-E(t,T)\omega(t)}$$

with

$$\begin{aligned} B(t,T) &= \frac{1}{a_r}\cdot(1-e^{-a_r(T-t)}), \\ E(t,T) &= \frac{b_{r\omega}}{a_r}\cdot\left(\frac{1-e^{a_\omega(T-t)}}{a_\omega}+\frac{e^{-a_\omega(T-t)}-e^{-a_r(T-t)}}{a_\omega-a_r}\right), \text{ and} \\ A(t,T) &= \int_t^T \frac{\sigma_r^2}{2}B^2(\tau,T)+\frac{\sigma_\omega^2}{2}E^2(t,T)-\theta_r(\tau)B(\tau,T)-\theta_\omega E(\tau,T)d\tau \end{aligned}$$

A proof of this statement can be found in Schmid et al. (2008). They also generalize the result of Proposition 5 to the pricing of defaultable zero-coupon bonds. We assume a fractional recovery of market value. Hence, there is a compensation in terms of equivalent defaultable bonds which have not defaulted yet, i.e. the recovery rate is expressed as a fraction of the market value of the defaulted bond just prior to default. By equivalent we mean bonds with the same maturity, quality and face value. This model was mainly developed by Duffie and Singleton (1999) and applied, e.g., by Schönbucher (2000). Then, the following proposition holds.

Proposition 6 Given the dynamics specified by equations (1)-(4), the value at time $t<\tau=\min(T,T^d)$, $P^d(t,T)=P^d(r,\omega,s,u,t,T)$, of a defaultable zero-coupon bond with maturity T is given by

$$\begin{aligned} P^d(t,T) &= e^{A^d(t,T)-B(t,T)r(t)-C(t,T)s(t)-D(t,T)u(t)-E^d(t,T)\omega(t)} \\ &= P(t,T)\cdot e^{A^*(t,T)-C(t,T)s(t)-D(t,T)u(t)+E^*(t,T)\omega(t)} \end{aligned}$$

where A(t,T), B(t,T), and E(t,T) are given in Proposition 5,

$$
\begin{aligned}
C(t,T) &= \frac{1}{a_s}\cdot(1-e^{-a_s(T-t)}),\\
D(t,T) &= \frac{b_{su}}{a_s}\cdot\left(\frac{1-e^{-a_u(T-t)}}{a_u}+\frac{e^{-a_u(T-t)}-e^{-a_s(T-t)}}{a_u-a_s}\right),\\
E^d(t,T) &= E(t,T)-E^*(t,T),\\
E^*(t,T) &= \frac{b_{s\omega}}{a_s}\cdot\left(\frac{1-e^{-a_\omega(T-t)}}{a_\omega}+\frac{e^{-a_\omega(T-t)}-e^{-a_s(T-t)}}{a_\omega-a_s}\right),\\
A^*(t,T) &= A^d(t,T)-A(t,T),
\end{aligned}
$$

and

$$
\begin{aligned}
A^d(t,T) = \int_t^T & \frac{1}{2}\sigma_s^2C^2(\tau,T)+\frac{1}{2}\sigma_u^2D^2(\tau,T)\\
&+\frac{1}{2}\sigma_r^2B^2(\tau,T)+\frac{1}{2}\sigma_\omega^2E^d(\tau,T)^2-\theta_sC(\tau,T)\\
&-\theta_uD(\tau,T)-\theta_r(\tau)B(\tau,T)-\theta_\omega E^d(\tau,T)d\tau\ .
\end{aligned}
$$

2 The Default Option Pricing Formulas

2.1 Bond Prices under Zero Recovery

Suppose we want to price an option on a defaultable zero-coupon bond, i.e. a so-called credit option. If the option is knocked out at default of the zero-coupon bond, the buyer of the option receives nothing. Hence, we can interpret the option as a defaultable investment with zero recovery. We show that we can determine the price of the option as the expected value of the promised cash flow at maturity of the option discounted at risky discount rates. As the recovery rate of the option is different from the recovery rate of the reference defaultable zero-coupon bond, the risky discount rates are not the same as in the case of the pricing of defaultable zero-coupon bonds. Hence, we have to find a short-rate credit spread s^{zero} describing the credit spread process of an obligor which is equivalent to the issuer of the zero-coupon bond (especially of the same quality) but with zero recovery rate. Therefore, for pricing credit derivatives such as credit options we need the following data for all maturities T > 0 :

- the default-free term structure of bond prices P(t,T),
- the defaultable term structure of bond prices P^d(t,T),
- the defaultable term structure of bond prices $P^{d,zero}$(t,T), under zero recovery.

Assumption 7 The zero-recovery short-rate spread s^{zero} is given by:

$$(1-z(t))\cdot s^{zero}(t)=s(t),\ 0\le t\le T^*, \qquad (5)$$

where s is the short-rate spread process defined in equation (4).

Proposition 8 Let Y be a $\mathcal{F}_T$-measurable random variable with $\mathbb{E}^Q[|Y|]<\infty$ for some q > 1. Under the zero-recovery assumption, i.e. under the assumption that the contingent claim is knocked out at default of the reference credit asset, and with the stochastic processes specified for r, ω, s, u, and s^{zero}, the price process, $V_{L,T}$,

$$V_{L,T}(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(l)dl} Y \cdot L(T) \mid \mathcal{G}_t \right], \; 0 \le t \le T,$$

is given by

$$V_{L,T}(t) = L(t) \cdot V_T(t)$$

where the adapted continuous process V_T is defined by

$$V_T(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r(l)+s^{zero}(l))dl} Y \mid \mathcal{F}_t \right], \; 0 \le t < T, \qquad (6)$$

and $V_T(t)=0$ for $t \ge T$. Hence, if there has been no default until time t, $V_{L,T}(t)$ must equal the expected value of riskless cashflows discounted at zero-recovery risky discount rates.

PROOF. See Schmid (2004), p. 230, with $\mathcal{F}_t$ substituted by $\mathcal{G}_t$ for the result under the enlarged filtration $\mathbb{G}$ and apply the martingale invariance property to show equation (6). □

Suppose we want to price a contingent claim that promises to pay off Y at maturity time T of the contingent claim, if the reference credit asset hasn't defaulted until then, and zero in case of a default. Then, the time t price of the reference credit asset, given there has been no default so far, depends on the stochastic processes r, w, s, and u. But in addition, the price of the contingent claim depends on s^{zero}, because discounting is done with $\exp(-\int_t^T r(l) + s^{zero}(l)dl)$.

In the following we assume that z(t) is a known constant, i.e. z(t)=z for all $0 \le t \le T^d$. Then, the dynamics of the zero-recovery short-rate spread are given by

$$ds^{zero}(t) = [\theta_s^{zero} + b_{su}^{zero} u(t) - b_{s\omega}^{zero} \omega(t) - a_s s^{zero}(t)]dt + \sigma_s^{zero} dW_s(t)$$

where

$$\theta_s^{zero} = \frac{\theta_s}{1-z}, \; b_{su}^{zero} = \frac{b_{su}}{1-z}, \; b_{s\omega}^{zero} = \frac{b_{s\omega}}{1-z}, \text{ and } \sigma_s^{zero} = \frac{\sigma_s}{1-z}.$$

Now we can calculate the zero-recovery zero-coupon bond prices

$$P^{d,zero}(t,T) = e^{A^{d,zero}(t,T) - B(t,T)r(t) - C^{zero}(t,T)s^{zero}(t) - D^{zero}(t,T)u(t) - E^{d,zero}(t,T)\omega(t)}$$

where $A^{d,zero}(t,T)$, $E^{d,zero}(t,T)$, $C^{d,zero}(t,T)$, and $D^{d,zero}(t,T)$ are given by the corresponding formulas for $A^d(t,T)$, $E^d(t,T)$, $C(t,T)$, and $D(t,T)$ with θ_s, b_{su}, $b_{s\omega}$, and σ_s substituted by θ_s^{zero}, b_{su}^{zero}, $b_{s\omega}^{zero}$ and σ_s^{zero}, respectively.

2.2 Default Put Options

A default (digital) put option is a credit derivative under which one party (the beneficiary) pays the other party (the guarantor) a fixed amount (lump-sum fee up-front). This is in exchange for the guarantor's promise to make a fixed or variable payment in the event of default in one or more reference assets to cover the full loss in default. As reference instruments for default put options we only consider defaultable zero-coupon bonds in this section, i.e. there is a payoff that is the difference between the face value and the market value (at default) of a reference credit asset (cash settlement). That is, the payoff at the time T^d of default is

$$Z(T^d) = 1 - P^d(T^d, T) = 1 - z \cdot P^d(T^d -, T) \ .$$

For a default digital put option the payoff is equal to 1 in case of a credit event before or at maturity. For the pricing of this derivative, let us first assume that the payoff takes place at maturity of the contract. Using equation (6), it is straightforward to show that the time t price of the default digital put option is given by

$$V_T^{ddp}(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(l)dl} H(T) \,|\, \mathcal{G}_t \right] = P(t,T) - L(t) \cdot P^{d,zero}(t,T)$$

with $P^{d,zero}(t,T)$ denoting the bond price under zero recovery.

Theorem 9 If the payoff takes place at default of the reference credit asset, the time t price of the default digital put option is given by

$$\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(l)dl} dH(u) \,|\, \mathcal{G}_t \right] = L(t) \cdot V_{T^d}^{ddp}(t)$$

with

$$\begin{aligned} V_{T^d}^{ddp}(t) &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u (r(l)+s^{zero}(l))dl} s^{zero}(u)du \,|\, \mathcal{F}_t \right] \\ &= \int_t^T \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^u (r(l)+s^{zero}(l))dl} s^{zero}(u) \,|\, \mathcal{F}_t \right] du \ . \end{aligned} \qquad (7)$$

PROOF. See Schmid (2004), p. 243, with $\mathcal{F}_t$ substituted by $\mathcal{G}_t$ for the result under $\mathbb{G}$ and apply the martingale invariance property to conclude with equation (7). □

The following theorem shows how to calculate the expected value in (7).

Theorem 10

$$
\begin{aligned}
\nu(r, s^{zero}, u, \omega, t, T) &:= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r(l)+s^{zero}(l))dl} s^{zero}(T) \mid \mathcal{F}_t \right] \\
&= P^{d,zero}(t,T) \cdot (F(t,T) + H(t,T)s^{zero}(t) + I(t,T)u(t) + J(t,T)\omega(t))
\end{aligned}
$$

with

$$
\begin{aligned}
H(t,T) &= e^{-a_s(T-t)}, \; J(t,T) = b_{s\omega}^{zero} \cdot \frac{e^{-a_s(T-t)} - e^{-a_\omega(T-t)}}{a_s - a_\omega}, \\
I(t,T) &= -b_{su}^{zero} \cdot \frac{e^{-a_s(T-t)} - e^{-a_u(T-t)}}{a_s - a_u}, \\
F(t,T) &= -\frac{1}{2}((\sigma_s^{zero} C(t,T))^2 + (\sigma_u D^{zero}(t,T))^2) \\
&\quad + \theta_\omega \cdot (E^{d,zero}(t,T) - E(t,T)) + \theta_s^{zero} C(t,T) \\
&\quad + \theta_u D^{zero}(t,T) - \int_t^T \sigma_\omega^2 E^{d,zero}(l,T) J(l,T) dl \, .
\end{aligned}
$$

PROOF. See Appendix. □

Finally, we consider the case of a default put option on a zero-coupon bond which replaces the difference to par if default is triggered.

Theorem 11 We assume that the underlying reference asset is a zero-coupon bond with maturity T* and that the default put has maturity T. Then, for t<T≤T* and replacement to the difference of par, we find the time t price of the default put to be

$$
V^{dp}(t) = L(t) \cdot (V_{T^d}^{dpp}(t) - P^d(t,T^*) + P^{d,*}(t,T,T^*)),
$$

where

$$
P^{d,*}(t,T,T^*) = e^{A^{d,*}(t,T,T^*) - B(t,T^*)r(t) - C^*(t,T,T^*)s(t) - D^*(t,T,T^*)u(t) - E^{d,*}(t,T,T^*)\omega(t)}
$$

with

$$
C^*(t,T,T^*) = \frac{1}{1-z} \cdot (C(t,T^*) - z e^{-a_s(T-t)} C(T,T^*)), \qquad (8)
$$

$$
\begin{aligned}
D^*(t,T,T^*) &= e^{-a_u(T-t)}D(T,T^*)+\frac{1}{1-z}D(t,T) \\
&\quad -b_{su}C(T,T^*)\left[\frac{e^{-a_s(T-t)}-e^{-a_u(T-t)}}{a_s-a_u}\right],
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
E^{d,*}(t,T,T^*) &= E(t,T^*)-e^{-a_\omega(T-t)}E^*(T,T^*)-\frac{1}{1-z}E^*(t,T) \\
&\quad +b_{s\omega}C(T,T^*)\left(\frac{e^{-a_\omega(T-t)}-e^{-a_s(T-t)}}{a_\omega-a_s}\right)
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
A^{d,*}(t,T,T^*) &= A^d(T,T^*)+\int_t^T \frac{\sigma_r^2}{2}B^2(\tau,T^*)+\frac{\sigma_s^2}{2}(C^*(\tau,T,T^*))^2 d\tau \\
&\quad +\int_t^T \frac{\sigma_u^2}{2}(D^*(\tau,T,T^*))^2+\frac{\sigma_\omega^2}{2}(E^{d,*}(\tau,T,T^*))^2 d\tau \\
&\quad -\int_t^T \theta_r(\tau)B(\tau,T^*)+\theta_s C^*(\tau,T,T^*)d\tau \\
&\quad -\int_t^T \theta_u D^*(\tau,T,T^*)+\theta_\omega E^{d,*}(\tau,T,T^*)d\tau .
\end{aligned}
$$

PROOF. See Appendix. □

Note, that $P^{d,*}(t,T,T)=P^{d,zero}(t,T)$. Also note, that the result of the previous theorem can be easily generalized to the case of coupon-paying bonds.

2.3. Default Swaps

A default swap is a swap under which one party (the beneficiary) pays the other party (the guarantor) regular fees, amounts that are based on a generic interest rate, called the default swap spread or the default swap rate. This is in exchange for the guarantor's promise to make a fixed or variable payment in the event of default in one or more reference assets to cover the full loss in default. As reference instruments we only consider defaultable (zero-coupon) bonds. In practice, default swap contracts differ in their specific default payments. We assume replacement of the difference to par which is currently market standard. A default swap on a defaultable coupon bond therefore pays off the difference between par and the post-default coupon bond price. There is only principal but no coupon protection. The pricing of a default swap consists of two problems. At origination there is no exchange of cashflows and we have to determine the default swap spread S that makes the market value of the default swap zero. After origination, the market value of the default swap will change due to changes in the underlying variables. So, given the default swap spread S, we have to determine the current market value of the default swap. We assume throughout that the credit swap counterparties (beneficiary and guarantor) are default-free.

2.3.1 The underlying reference credit asset is a defaultable zero-coupon bond

We assume that the underlying reference credit asset is a defaultable zero-coupon bond with maturity T* and that there has been no credit event until time t. In case of a credit default swap with maturity T≤T* there are regular payments S (the credit swap spread) instead of an up-front fee $V^{dp}(0)$. The value of paying $V^{dp}(0)$ at the origination of the credit-default put option must be the same as paying S at some predefined times $t \le t_1 \le \ldots \le t_m = T$ until a default happens. Hence,

$$V^{dp}(0) = S \cdot \sum_{i=1}^{m} P^{d,zero}(0, t_i) \ .$$

This is equivalent to a credit swap spread of

$$S = \frac{V^{dp}(0)}{\sum_{i=1}^{m} P^{d,zero}(0, t_i)} \ .$$

2.3.2 The underlying reference credit asset is a defaultable coupon bond

In the following we assume that the underlying reference asset is a defaultable coupon bond. In addition, we assume that there has been no credit event until time t:

(1) Default put options and replacement of the difference to par:

The reference credit asset is a coupon bond with maturity T* and discrete coupon payments c_i occurring at dates $t \le \tau_1 \le \cdots \le \tau_n = T^*$. Then, the pricing argument for the default put option with maturity T≤T* is exactly the same as in the case of the zero-coupon bond, and we get

$$V_c^{dp}(t) = V_{T^d}^{ddp}(t) - P_c^d(t, T^*) + P_c^{d,*}(t, T, T^*) \ ,$$

where

$$P_c^d(t, T^*) = \sum_{i=1}^{n} c_i P^d(t, \tau_i) + P^d(t, T^*)$$

and

$$P_c^{d,*}(t, T, T^*) = \sum_{i=1}^{n} c_i P^{d,*}(t, T, \tau_i) + P^{d,*}(t, T, T^*)$$

with

$$P^{d,*}(t, T, \tau_i) := P^{d,zero}(t, \tau_i) \ \text{if} \ \tau_i \le T, \ \ i = 1, \ldots, n.$$

(2) Default swaps:

The credit swap spread can be calculated by the same argument as in the case of zero-coupon bonds. Hence, if there are regular payments S_c at some predefined times $t \le t_1 \le \cdots \le t_m = T$ until a default happens, S_c is given by

$$S_c = \frac{V_c^{dp}(0)}{\sum_{i=1}^{m} P^{d,zero}(0, t_i)} .$$

3 Data and Parameter Estimation

3.1 Estimation of the Parameters of the Underlying Processes

We estimate the parameters of the underlying processes from observable time series of defaultable and non-defaultable EUR zero rates. All parameters are estimated using Kalman Filter methodologies as suggested, e.g., by Schmid (2004). As our main data source we use Bloomberg. We use monthly data from March 29, 2002, until January 31, 2007. However, as can be seen from Graphs F1 and F2, due to the credit crisis spread levels have exploded and GDP growth rates went dramatically down from the middle of 2007. Therefore, we update the parameters by including additional data up to 30th June 2007, 31st January 2008, 30th June 2008, and 30th September 2008. This allows us to investigate the impact of the 2007/2008 credit crunch on the parameter estimates.

For the calibration of the non-defaultable EUR short rates we use monthly observations of the EUR zero rates with maturities between 3 months and 30 years (Bloomberg tickers F96003M Index … F960030Y Index). The parameter estimates for the different time horizons can be found in Table D1. For the calibration of the credit spread processes, we use monthly observations of the EUR AA, and BBB credit spreads (Bloomberg tickers: C6673M Index … C66730Y Index and C6733M Index … C67330Y Index, resp., see also Graphs F1 and F2). The parameter estimates are summarized in Tables E1 and E2. As can be seen in these tables the volatilities of the spreads strictly increased from January 2007 to September 2008. In addition to the interest-rate and spread data, we consider quarterly growth rates of the EUR GDP rates (Bloomberg ticker EUGNEMU Index – for results see Table C1). As there is no monthly data observable, we generate the missing data using a linear interpolation with a three-quarter time lag consistent to the market standard. As can be seen from Table D1 (parameter b_{rw}) and Tables E1 and E2 (parameter b_{sw}) the influence of the GDP decreased during the year 2008. This indicates that the behaviour of interest rates and spreads was rather influenced by the credit and liquidity effects of the financial crisis than by the general state of the economy. In late 2008 the effect of an economic crisis following the credit and liquidity crisis becomes visible in increasing values of b_{rw} and b_{sw} (rating class AA). For all our estimations we use the software packages R and S-PLUS Finmetrics. The results of the parameter estimates are summarized in the Appendix. As the process u is unobservable, we set $b_{su}=1$. More details with respect to parameter estimation techniques as well as in- and out-of-sample tests of the model can be found in Schmid et al. (2008).

3.2 Calibrating the Model to Market Prices of Credit Default Swaps

Finally we want to calibrate our model to observable market prices (mid closing prices) of credit default swaps. To ensure high liquidity of the instruments we pick constituents of the iTraxx Europe Series 8 with a maturity of 5 years. As examples we show here CDS contracts on the AA and BBB rated issuers BBVA and Deutsche Lufthansa, respectively. The daily closing mid spreads of these credit default swaps from 31st January 2007 until 30th September 2008 can be seen in Figures G3 and G4. The observable spreads at our parameter estimation update dates are summarized in Table 3.2.1.

Table 3.2.1: Closing Mid CDS Spreads (in BPs) of BBVA and Lufthansa
Source UBS Delta and MarkIt

	31-Jan-2007	29-Jun-2007	31-Jan-2008	30-Jun-2008	30-Sep-2008
BBVA:	8.9	12	74	73.1	130
Lufthansa:	40.5	47	88	183	168

Based on the historical parameter estimations of the underlying processes we estimate the recovery rate parameters such that the observed market prices equal the theoretical model prices at the 31st January 2007. We assume that these recovery rates are constant through time, and the estimates are 80.39% for BBVA and 75.45% for Lufthansa. Note, that these estimates are for recovery rates expressed as percentages of market values of bonds prior to default. These estimates correspond to much lower recovery values if expressed as a percentage of par (which is in line with recovery rates usually assumed in the market). If we priced the CDS contracts at our other valuation dates (29-Jun-2007, 31-Jan-2008, 30-Jun-2008, 30-Sep-2008) based on these recovery rate assumptions as well as the parameter estimates in Appendix C-E we would obviously see deviations of the model prices from the observable market prices. These deviations go back to the fact that we would use average historical data for the estimation of the parameters as well as the very well known CDS basis between bond and CDS markets. Among other factors the different liquidity in these two markets is one of the most important explanations for the CDS basis. Especially during the credit crunch liquidity became a crucial factor. Therefore, in order to explain CDS prices during this phase of market dislocation and bond illiquidity, we suggest to adjust the parameter θ_u in the process u. As in a wider interpretation the process u can be understood as an uncertainty and illiquidity factor aggregating idiosyncratic default and liquidity risk, it is influencing default as well as liquidity components in bond and CDS spreads heavily. Note, that this parameter adjustment was not necessary at the beginning of 2007 where we could directly imply a meaningful recovery rate estimate from the market CDS spread and our historically estimated parameters. In January 2007 the difference in the liquidity of the two markets was not as present as from mid 2007 onwards when markets became more difficult. The following table shows the values of θ_u before and after adjustment. Model prices based on the adjusted parameters fully match observed market prices. The model clearly indicates that the uncertainty and illiquidity in the market, represented by the adjusted parameter θ_u dramatically increased in January 2008 and still is at a much higher level than in June 2007.

Table 3.2.2: Parameter θ_u before (upper values) and after adjustment

	29-Jun-2007	31-Jan-2008	30-Jun-2008	30-Sep-2008
BBVA:	0.000524 0.011715	0.001737 0.851452	0.000773 0.053368	0.000678 0.063685
Lufthansa:	0.002824 0.049289	0.001669 0.118384	0.003698 0.076531	0.002889 0.081613

4 Summary and Conclusion

We developed pricing formulas for credit default options and swaps based on the extended Schmid and Zagst defaultable term-structure model. Hereby, we related credit spreads to the business cycle and assumed that the spread between a defaultable and a non-defaultable bond is considerably driven by an uncertainty index modeling the idiosyncratic default risk (and liquidity) of a counterparty. We fitted our model to market prices of credit default swaps and calculated recovery rates implied by these market prices. As we apply average historical data for estimating the parameters of the underlying processes, we used the uncertainty index (parameter θ_u) to adjust the model to market prices through time. This also adjusts for the different liquidity in the bond and CDS markets. Increasing credit risk is indicated by increasing volatilities of the spreads as well as an increase in the level of the uncertainty index. The model also shows a decreasing influence of the GDP on interest rates and credit spreads during the year 2008.

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APPENDIX

A) PROOF Theorem 10:

For ease of notation we omit the superscript zero in this proof. According to the theorem of Feynman-Kac (see, e.g., Duffie (1992), p. 241-244, or Zagst (2002), p. 38-41), v is the solution of the following PDE:

$$\begin{aligned} 0 \quad = \quad & \frac{1}{2}(\sigma_s^2 \nu_{ss} + \sigma_u^2 \nu_{uu} + \sigma_\omega^2 \nu_{\omega\omega} + \sigma_r^2 \nu_{rr}) + (\theta_r(t) + b_r\omega - a_r r)\nu_r \\ & +(\theta_\omega - a_\omega \omega)\nu_\omega + (\theta_u - a_u u)\nu_u + (\theta_s + b_{su} u - b_{s\omega}\omega - a_s s)\nu_s \\ & -(r+s)\nu + \nu_t \end{aligned}$$

under the condition v(r, s, u, w, T, T) = s. If

$$\begin{aligned} \nu(r,s,u,\omega,t,T) \quad &= \quad P^d(t,T)(F(t,T) + G(t,T)r + H(t,T)s + I(t,T)u + J(t,T)\omega) \\ &= \quad e^{A^d(t,T) - B(t,T)r - E^d(t,T)\omega - C(t,T)s - D(t,T)u} \\ & \quad \cdot (F(t,T) + G(t,T)r + H(t,T)s + I(t,T)u + J(t,T)\omega), \end{aligned}$$

then

$$\begin{aligned} 0 \quad = \quad & \frac{1}{2}(\sigma_s^2 C^2 + \sigma_u^2 D^2 + \sigma_\omega^2 (E^d)^2 + \sigma_r^2 B^2)(F + Gr + Hs + Iu + J\omega) \\ & +(-\sigma_s^2 CH - \sigma_u^2 DI - \sigma_\omega^2 E^d J - \sigma_r^2 BG) \\ & +(\theta_r(t) + b_r\omega - a_r r)(-B(F + Gr + Hs + Iu + J\omega) + G)) \\ & +(\theta_\omega - a_\omega\omega)(-E^d(F + Gr + Hs + Iu + J\omega) + J) \\ & +(\theta_s + b_{su}u - b_{s\omega}\omega - a_s s)(-C(F + Gr + Hs + Iu + J\omega) + H) \\ & +(\theta_u - a_u u)(-D(F + Gr + Hs + Iu + J\omega) + I) \\ & -(r+s)(F + Gr + Hs + Iu + J\omega) \\ & +(F + Gr + Hs + Iu + J\omega)(A_t^d - C_t s - D_t u - E_t^d\omega - B_t r) \\ & +(F_t + G_t r + H_t s + I_t u + J_t\omega) \ . \end{aligned}$$

This reduces to

$$
\begin{aligned}
0 &= -\sigma_s^2 CH - \sigma_u^2 DI - \sigma_\omega^2 E^d J - \sigma_r^2 BG \\
&\quad + G\theta_r(t) + J\theta_\omega + H\theta_s + I\theta_u + F_t + r(-a_r G + G_t) \\
&\quad + \omega(-a_\omega J + J_t + b_r G - b_{s\omega} H) + s(-a_s H + H_t) \\
&\quad + u(-a_u I + I_t + b_{su} H)
\end{aligned}
$$

under the boundary conditions

$$
G(T,T)=0,\ F(T,T)=0,\ H(T,T)=1,\ I(T,T)=0,\ J(T,T)=0\ .
$$

Therefore, we have to solve the following system of linear equations:

$$
\begin{aligned}
G_t &= a_r G,\ J_t = a_\omega J - b_r G + b_{s\omega} H,\ H_t = a_s H,\ I_t = a_u I - b_{su} H, \\
F_t &= \sigma_s^2 CH + \sigma_u^2 DI + \sigma_\omega^2 E^d J + \sigma_r^2 BG - G\theta_r(t) - J\theta_\omega - H\theta_s - I\theta_u .
\end{aligned}
$$

Applying the transformation τ=T-t we get:

$$
\begin{aligned}
G(t,T) &= 0,\ H(t,T) = e^{-a_s(T-t)}, \\
J(t,T) &= e^{-a_\omega(T-t)} \int_0^{T-t} -e^{a_\omega l} b_{s\omega} e^{-a_s l} dl = b_{s\omega} \cdot \frac{e^{-a_s(T-t)} - e^{-a_\omega(T-t)}}{a_s - a_\omega} \\
I(t,T) &= e^{-a_u(T-t)} \int_0^{T-t} e^{a_u l} b_{su} e^{-a_s l} dl = -b_{su} \cdot \frac{e^{-a_s(T-t)} - e^{-a_u(T-t)}}{a_s - a_u} \\
F(t,T) &= \int_t^T -\sigma_s^2 C(l,T)H(l,T) - \sigma_u^2 D(l,T)I(l,T) \\
&\quad -\sigma_\omega^2 E^d(l,T)J(l,T) - \sigma_r^2 B(l,T)G(l,T) \\
&\quad + G(l,T)\theta_r(l) + J(l,T)\theta_\omega + H(l,T)\theta_s + I(l,T)\theta_u dl\ .
\end{aligned}
$$

Using

$$
-C_t = H,\ -D_t = I,\ -(E^d - E)_t = J
$$

we finally get

$$
\begin{aligned}
F(t,T) &= \frac{1}{2}(\sigma_s^2 C(t,T)^2 + \sigma_u^2 D(t,T)^2) + \theta_\omega (E^d - E)(t,T) \\
&\quad + \theta_s C(t,T) + \theta_u D(t,T) - \int_t^T \sigma_\omega^2 E^d(l,T)J(l,T)dl\ .
\end{aligned}
$$

□

B) PROOF Theorem 11:

Let Z describe the payoff of the underlying bond upon default. Then, we get:

$$
\begin{aligned}
V^{dp}(t) &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(l)dl}(1-Z(u))dH(u) \mid \mathcal{G}_t\right] \\
&= L(t)\cdot V^{ddp}_{T^d}(t) - \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(l)dl} Z(u)dH(u) \mid \mathcal{G}_t\right] \\
&= L(t)\cdot V^{ddp}_{T^d}(t) - \mathbb{E}^{\mathbb{Q}}\left[\int_t^{T^*} e^{-\int_t^u r(l)dl}(1-Z(u))dH(u) \mid \mathcal{G}_t\right] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(l)dl}\int_T^{T^*} e^{-\int_T^u r(l)dl} Z(u)dH(u) \mid \mathcal{G}_t\right] \\
&= L(t)\cdot (V^{ddp}_{T^d}(t) - P^d(t,T^*) + P^{d,zero}(t,T^*)) \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(l)dl} E^Q\left[\int_T^{T^*} e^{-\int_T^u r(l)dl} Z(u)dH(u) \mid \mathcal{G}_T\right] \mid \mathcal{G}_t\right] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(l)dl} E^Q\left[e^{-\int_T^{T^*} r(l)dl} L(T^*) \mid \mathcal{G}_T\right] \mid \mathcal{G}_t\right] \\
&\quad - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(l)dl} E^Q\left[e^{-\int_T^{T^*} r(l)dl} L(T^*) \mid \mathcal{G}_T\right] \mid \mathcal{G}_t\right] \\
&= L(t)\cdot (V^{ddp}_{T^d}(t) - P^d(t,T^*) + P^{d,zero}(t,T^*)) \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(l)dl} P^d(T,T^*)\cdot L(T) \mid \mathcal{G}_t\right] \\
&\quad - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(l)dl} P^{d,zero}(T,T^*)\cdot L(T) \mid \mathcal{G}_t\right].
\end{aligned}
$$

By applying Proposition 8 we get

$$
\begin{aligned}
V^{dp}(t) &= L(t)\cdot(V^{ddp}_{T^d}(t) - P^d(t,T^*) + P^{d,zero}(t,T^*)) \\
&\quad + L(t)\cdot\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r(l)+s^{zero}(l))dl} P^d(T,T^*) \mid \mathcal{F}_t \right] \\
&\quad - L(t)\cdot\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r(l)+s^{zero}(l))dl} P^{d,zero}(T,T^*) \mid \mathcal{F}_t \right]
\end{aligned}
$$

which is equivalent to

$$
V^{dp}(t) = L(t)\cdot(V^{ddp}_{T^d}(t) - P^d(t,T^*) + P^{d,*}(t,T,T^*)) \;,
$$

where

$$
P^{d,*}(t,T,T^*) := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r(l)+s^{zero}(l))dl} P^d(T,T^*) \mid \mathcal{F}_t \right].
$$

To calculate $P^{d,*}$ we assume that there exists a solution of the type

$$
P^{d,*}(t,T,T^*) = e^{A^{d,*}(t,T,T^*) - B^*(t,T,T^*)r(t) - C^*(t,T,T^*)s(t) - D^*(t,T,T^*)u(t) - E^{d,*}(t,T,T^*)\omega(t)}
$$

and apply the theorem of Feynman-Kac to derive a system of corresponding linear equations:

$$
\begin{aligned}
a_r B^*(t,T,T^*) - B^*_t(t,T,T^*) - 1 &= 0 \\
a_s C^*(t,T,T^*) - C^*_t(t,T,T^*) - \frac{1}{1-z} &= 0 \\
a_u D^*(t,T,T^*) - b_{su} C^*(t,T,T^*) - D^*_t(t,T,T^*) &= 0 \\
a_\omega E^{d,*}(t,T,T^*) - b_{r\omega} B^*(t,T,T^*) + b_{s\omega} C^*(t,T,T^*) - E^{d,*}_t(t,T,T^*) &= 0
\end{aligned}
$$

and

$$
\begin{aligned}
-A^{d,*}_t(t,T,T^*) &= \frac{1}{2}(\sigma_r^2 (B^*(t,T,T^*))^2 + \sigma_s^2 (C^*(t,T,T^*))^2 \\
&\quad + \sigma_u^2 (D^*(t,T,T^*))^2 + \sigma_\omega^2 (E^{d,*}(t,T,T^*))^2 \\
&\quad - \theta_r(t) B^*(t,T,T^*) - \theta_s C^*(t,T,T^*) \\
&\quad - \theta_u D^*(t,T,T^*) - \theta_\omega E^{d,*}(t,T,T^*)
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
A^{d,*}(T,T,T^*) &= A^d(T,T^*),\ B^*(T,T,T^*) = B(T,T^*),\\
C^*(T,T,T^*) &= C(T,T^*),\ D^*(T,T,T^*) = D(T,T^*),\\
E^{d,*}(T,T,T^*) &= E^d(T,T^*)\ .
\end{aligned}
$$

- Solving for B*:

$$B^*(t,T,T^*) = e^{-a_r(T-t)}\left(\frac{1}{a_r}\cdot\left(1-e^{-a_r(T^*-T)}\right)+\int_0^{T-t} e^{a_r l}dl\right) = B(t,T^*)\ .$$

- Solving for C*: Let

$$\hat{C}(t,T,T^*) = C^*(t,T,T^*)\cdot(1-z)\ .$$

Then

$$a_s C^*(t,T,T^*) - C_t^*(t,T,T^*) - \frac{1}{1-z} = 0$$

is equivalent to

$$a_s\hat{C}(t,T,T^*) - \hat{C}_t(t,T,T^*) - 1 = 0\ .$$

Using the theorem of Feynman-Kac, we can easily see from Proposition 6 that

$$C(t,T^*) = a_s C_t(t,T^*) - 1\ .$$

Hence, using Equation (8),

$$
\begin{aligned}
a_s\hat{C}(t,T,T^*) - \hat{C}_t(t,T,T^*) &= a_s C(t,T^*) - a_s z e^{-a_s(T-t)}C(T,T^*)\\
&\quad - C_t(t,T^*) + a_s z e^{-a_s(T-t)}C(T,T^*)\\
&= a_s C(t,T^*) - C_t(t,T^*) = 1\ .
\end{aligned}
$$

In addition,

$$\hat{C}(T,T,T^*) = C(T,T^*)(1-z) = C^*(T,T,T^*)(1-z)$$

which is equivalent to

$$C^*(T,T,T^*) = C(T,T^*)\ .$$

- Solving for D*: Using Equation (9):

$$\begin{aligned}
a_u D^*(t,T,T^*) - D_t^*(t,T,T^*) &= a_u e^{-a_u(T-t)} D(T,T^*) + \frac{a_u}{1-z} D(t,T) \\
&\quad - a_u b_{su} C(T,T^*) \left[\frac{e^{-a_s(T-t)} - e^{-a_u(T-t)}}{a_s - a_u} \right] \\
&\quad - a_u e^{-a_u(T-t)} D(T,T^*) - \frac{1}{1-z} D_t(t,T) \\
&\quad + b_{su} C(T,T^*) \left[\frac{a_s e^{-a_s(T-t)} - a_u e^{-a_u(T-t)}}{a_s - a_u} \right] \\
&= \frac{1}{1-z}(a_u D(t,T) - D_t(t,T)) + b_{su} C(T,T^*) e^{-a_s(T-t)} \ .
\end{aligned}$$

Again, using the theorem of Feynman-Kac, it can be easily seen from Proposition 6 that

$$D_t(t,T) = a_u D(t,T) - b_{su} C(t,T) \ .$$

Hence,

$$\begin{aligned}
a_u D^*(t,T,T^*) - D_t^*(t,T,T^*) &= \frac{b_{su}}{1-z}(C(t,T) + (1-z)C(T,T^*)e^{-a_s(T-t)}) \\
&= \frac{b_{su}}{1-z}(C(t,T^*) - zC(T,T^*)e^{-a_s(T-t)}) \\
&= b_{su} C^*(t,T,T^*) \ .
\end{aligned}$$

In addition,

$$D^*(T,T,T^*) = D(T,T^*) + \frac{1}{1-z} D(T,T) = D(T,T^*) \ .$$

- Solving for $E^{d,*}$: Using Equation (10):

$$
\begin{aligned}
a_\omega E^{d,*}(t,T,T^*) - E_t^{d,*}(t,T,T^*) &= a_\omega E(t,T^*) - a_\omega e^{-a_\omega(T-t)} E^*(T,T^*) - \frac{a_\omega}{1-z} E^*(t,T) \\
&\quad + b_{s\omega} C(T,T^*) \left(\frac{a_\omega e^{-a_\omega(T-t)} - a_\omega e^{-a_s(T-t)}}{a_\omega - a_s} \right) \\
&\quad - E_t(t,T^*) + a_\omega e^{-a_\omega(T-t)} E^*(T,T^*) + \frac{1}{1-z} E_t^*(t,T) \\
&\quad - b_{s\omega} C(T,T^*) \left(\frac{a_\omega e^{-a_\omega(T-t)} - a_s e^{-a_s(T-t)}}{a_\omega - a_s} \right) \\
&= a_\omega E(t,T^*) - E_t(t,T^*) - b_{s\omega} C(T,T^*) e^{-a_s(T-t)} \\
&\quad + \frac{1}{1-z} (a_\omega E^d(t,T) - E_t^d(t,T) - (a_\omega E(t,T) - E_t(t,T))) \; .
\end{aligned}
$$

Using the theorem om Feynman-Kac, we know from Propositions 5 and 6 that

$$
E_t(t,T) = a_\omega E(t,T) - b_{r\omega} B(t,T)
$$

and

$$
E_t^d(t,T) = a_\omega E^d(t,T) - b_{r\omega} B(t,T) + b_{s\omega} C(t,T) \; .
$$

Hence,

$$
\begin{aligned}
a_\omega E^{d,*}(t,T,T^*) - E_t^{d,*}(t,T,T^*) &= b_{r\omega} B(t,T^*) - \frac{b_{s\omega}}{1-z} C(t,T) \\
&\quad - b_{s\omega} C(T,T^*) e^{-a_s(T-t)} \\
&= b_{r\omega} B(t,T^*) - \frac{b_{s\omega}}{1-z} C(t,T) \\
&\quad - b_{s\omega} C(T,T^*) e^{-a_s(T-t)} \\
&= b_{r\omega} B(t,T^*) - \frac{b_{s\omega}}{1-z} C(t,T^*) \\
&\quad - \frac{b_{s\omega} z}{1-z} C(T,T^*) e^{-a_s(T-t)}
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
a_\omega E^{d,*}(t,T,T^*) - E_t^{d,*}(t,T,T^*) &= b_{r\omega} B(t,T^*) - b_{s\omega} C^*(t,T,T^*) \\
&= b_{r\omega} B^*(t,T,T^*) - b_{s\omega} C^*(t,T,T^*) \; .
\end{aligned}
$$

In addition,

$$\begin{aligned} E^{d,*}(T,T,T^*) &= E(T,T^*) - E^*(T,T^*) - \frac{1}{1-z} E^*(T,T) \\ &= E(T,T^*) - E^*(T,T^*) = E^d(T,T^*) \ . \end{aligned}$$

- Solving for $A^{d,*}$:

$$\begin{aligned} A^{d,*}(t,T,T^*) - A^{d,*}(T,T,T^*) &= A^{d,*}(t,T,T^*) - A^d(T,T) \\ &= -\int_t^T A_t(\tau,T,T^*)d\tau \end{aligned}$$

and the solution follows from simple integration. □

C) Tables of Parameter Estimates – GDP Rates

Table C.1: EUR GDP Parameters

	31-Jan-2007	29-Jun-2007	31-Jan-2008	30-Jun-2008	30-Sep-2008
a_{ω}	0.9095	0.7609	0.9885	0.8345	0.8122
θ_{ω}	0.003313	0.003367	0.003487	0.0036	0.003989
σ_{ω}	0.003657	0.003596	0.003533	0.00354	0.003851
|Log-Likelihood|	518.8	547.8	588.1	615.5	621.6

D) Tables of Parameter Estimates – Non-Defaultable Short Rate Parameters

Table D.1: EUR Short Rate Parameters

	31-Jan-2007	29-Jun-2007	31-Jan-2008	30-Jun-2008	30-Sep-2008
a_r	0.4431	0.2463	0.2487	0.4102	0.3166
$b_{r\omega}$	0.1559	0.1275	0.1044	0.05151	0.1013
σ_r	0.01005	0.007876	0.007944	0.009699	0.008729
|Log-Likelihood|	3286	2721	2896	3832	3923

E) Tables of Parameter Estimates – Short Rate Credit Spreads

Table E.1: EUR AA Credit Spread Parameters

	31-Jan-2007	29-Jun-2007	31-Jan-2008	30-Jun-2008	30-Sep-2008
a_s	0.8312	0.3287	0.4251	0.2985	0.4933
θ_s	0.001252	0.001277	0.0001275	1.465e-005	0.00241
σ_s	0.0008983	0.0008872	0.00151	0.002337	0.003182
$b_{s\varpi}$	0.0004295	0.02813	0.01001	0.004741	0.1041
b_{su}	1	1	1	1	1
a_u	0.3668	0.9653	0.0593	0.4404	0.5592
θ_u	0.0002971	0.0005242	0.001737	0.0007725	0.0006778
σ_u	0.000469	0.001533	0.02112	0.005093	0.008816
|Log-Likelihood|	1887	2097	1591	2167	2000

Table E.2: EUR BBB Credit Spread Parameter

	31-Jan-2007	29-Jun-2007	31-Jan-2008	30-Jun-2008	30-Sep-2008
a_s	0.6496	0.6436	0.763	0.03331	0.4193
θ_s	0.004823	0.001222	0.005086	0.000454	5.349e-005
σ_s	0.001779	0.00185	0.002421	0.002973	0.004298
$b_{s\varpi}$	0.2183	0.1203	0.4374	0.2906	0.09913
b_{su}	1	1	1	1	1
a_u	0.3245	0.9604	0.3051	0.9355	0.2438
θ_u	0.0001652	0.002824	0.001669	0.003698	0.002889
σ_u	0.006776	0.01073	0.01856	0.01037	0.01174
|Log-Likelihood|	973.7	1494	1424	1162	1539

F) Time Series of Growth Rates of EUR GDP vs AA and BBB Credit Spreads

Figure F.1: Growth Rates EUR GDP and EUR AA Credit Spreads, Source: Bloomberg

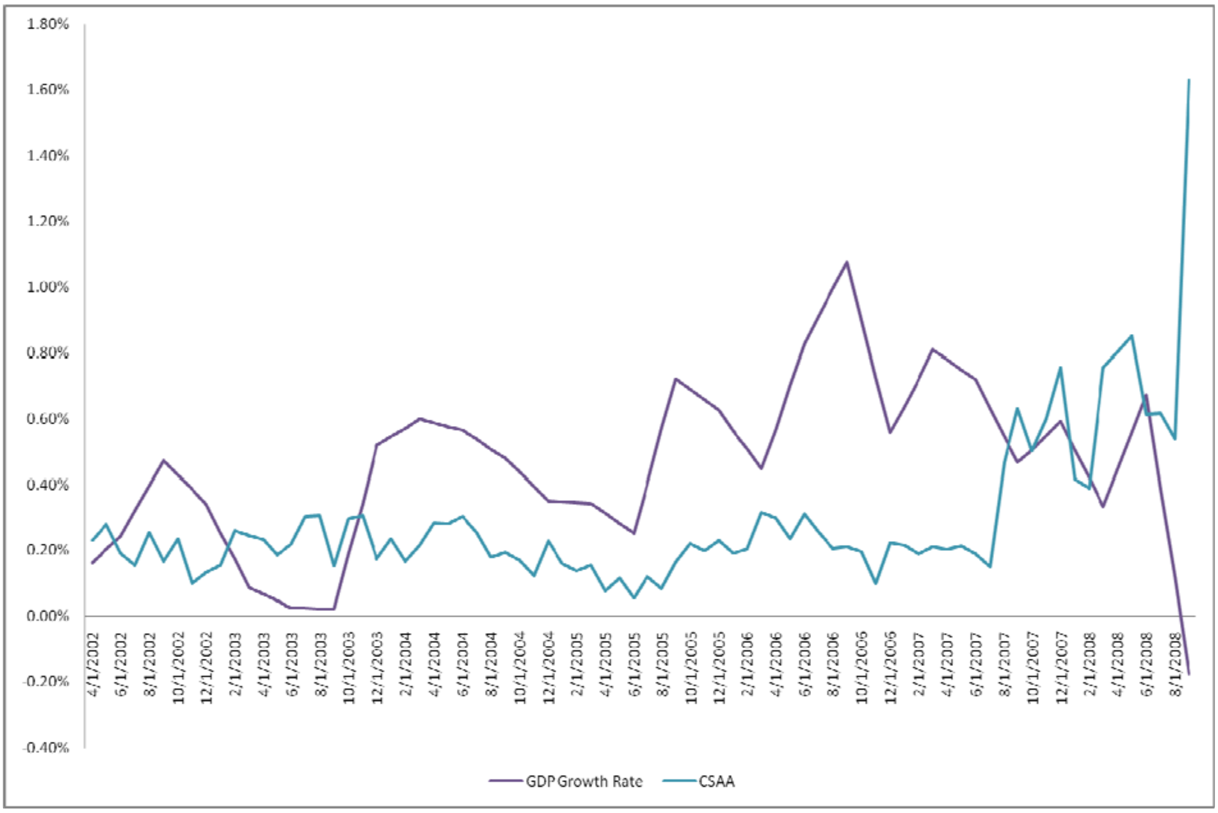
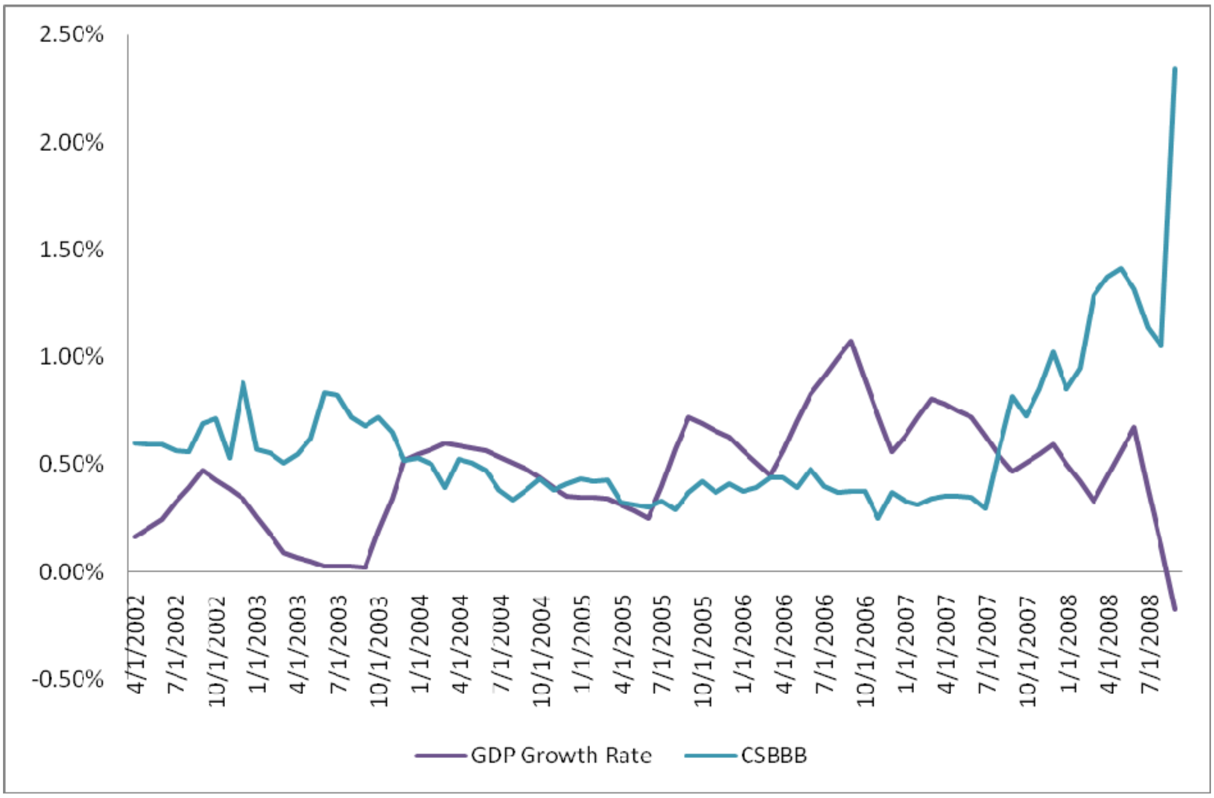


Figure F.2: Growth Rates EUR GDP and EUR BBB Credit Spreads: Source Bloomberg



G) Asset Details and Spread Histories of CDS contracts

Figure G.1: Asset Details of BBVA: Source UBS Delta

Banco Bilbao Vizcaya Argentaria, Sociedad Anonima (SEN)

Definition | Issuer Securities

Identifiers	
RED ID	EF2985
Ticker	BBVSM
Issuer ID	26897
Asset ID	CDS00468

Ratings	
Moody's	Aa1
S&P	AA
Composite	AA3
iBoxx	-

Sectors	Broad	Intermediate	Fine
UBS	Financials	Financial - Banks	Banking & Finance
iBoxx	Financials	Financials	Bank
US	-	-	-

Reference Obligations

Issuer	ISIN	Debt Seniority	Coupon	Maturity Date
BANCO BILBAO VIZCAYA ARGENTARIA SA	XS0250172003	UNS	4	22 Apr 2013
BANCO BILBAO VIZCAYA ARGENTARIA SA	ES0313211130	UNS	4.07	26 Feb 2009
BANCO BILBAO VIZCAYA ARGENTARIA SA	XS0086908802	UNS	5.25	25 May 2010

Figure G.2: Asset Details of Deutsche Lufthansa: Source UBS Delta

Deutsche Lufthansa Aktiengesellschaft (SEN)

Definition | Issuer Securities

Identifiers	
RED ID	2H66B7
Ticker	LUFTHA
Issuer ID	43354
Asset ID	CDS00635

Ratings	
Moody's	Baa3
S&P	BBB
Composite	BBB3
iBoxx	-

Sectors	Broad	Intermediate	Fine
UBS	Industrials	Industrials - Engineering	Aerospace & Defense
iBoxx	Corporates	Consumer Services	Travel & Leisure
US	-	-	-

Reference Obligations

Issuer	ISIN	Debt Seniority	Coupon	Maturity Date
DEUTSCHE LUFTHANSA AG	DE000A0JQA39	UNS	4.625	06 May 2013

Figure G.3: BBVA 5 Year CDS Spreads: Source UBS Delta, MarkIt

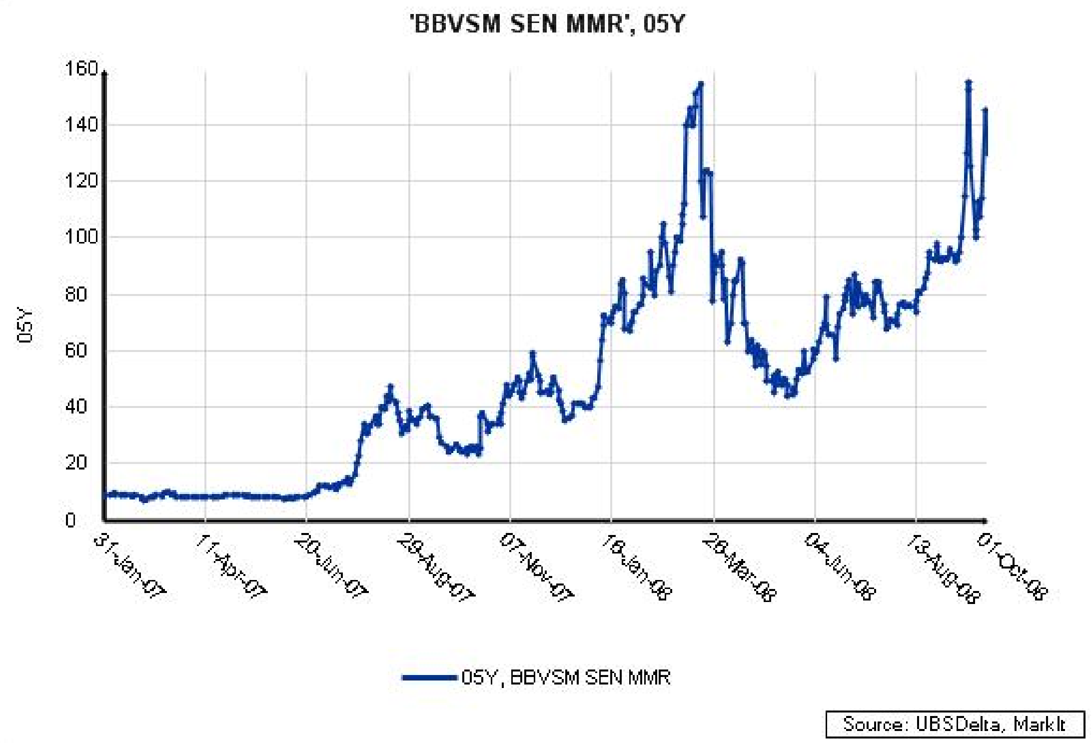


Figure G.4: Lufthansa 5 Year CDS Spreads: Source UBS Delta, MarkIt