risklab germany | advanced asset management solutions

working paper

The Normal inverse Gaussian distribution for synthetic CDO pricing.

Anna Kalemanova
Bernd Schmid
Ralf Werner

August 2005

risklab germany GmbH

Nymphenburger Str. 112-116

80636 München

Tel. +49.89.1220 7750

05-03

The Normal Inverse Gaussian Distribution for Synthetic CDO Pricing

Anna Kalemanova* Bernd Schmid† Ralf Werner‡

Abstract

This paper presents an extension of the popular Large Homogeneous Portfolio (LHP) approach to the pricing of CDOs. LHP (which has already become a standard model in practice) assumes a flat default correlation structure over the reference credit portfolio and models default using a one factor Gaussian copula. However, this model fails to fit the prices of different CDO tranches simultaneously which leads to the well known implied correlation smile. Many researchers explain this phenomenon with the lack of tail dependence and propose to use a Student t copula. Incorporating the effect of tail dependence into the one factor portfolio credit model yields significant pricing improvement. However, the computation time increases dramatically as the Student t distribution is not stable under convolution. This makes it impossible to use the model for computationally intensive applications such as the determination of the optimal asset allocation in an investor's portfolio over different asset classes including CDOs. We present a modification of the LHP model replacing the Student t distribution with the Normal inverse Gaussian (NIG) distribution. We compare the properties of our new model with those of the Gaussian and the double t copulas. The employment of the NIG distribution does not only speed up the computation time significantly but also brings more flexibility into the dependence structure.

*risklab germany, Nymphenburger Str. 112-116, 80636 Munich, Germany, anna.kalemanova@risklab.de

†Algorithmics Incorporated, New Broad Street House, 35 New Broad Street, London EC2M 1NH, UK, bernd.schmid@algorithmics.com

‡Allianz, GRC Risk Methodology, Königinstr. 28, 80802 Munich, Germany, ralf.werner@allianz.de

Introduction

The calculation of loss distributions of the portfolio of reference instruments over different time horizons is the central problem of pricing synthetic CDOs. The factor copula approach for modelling correlated defaults has become very popular. Unfortunately, computationally intensive Monte Carlo simulation techniques have to be used if the correlation structure is assumed to be completely general. However, the concept of conditional independence yields substantial simplification: If it is assumed that defaults of different titles in the credit portfolio are independent conditional on a common market factor it is much simpler to compute the aggregate portfolio loss distributions for different time horizons. The factor (or conditional independence) approach allows to use semi-analytic computation techniques avoiding time consuming Monte Carlo simulations. Examples are the approaches described by Gregory and Laurent [8] who use fast Fourier transformation techniques, as well as Hull and White [7] and Andersen, Sidenius and Basu [2] who apply an iterative numerical procedure to build up the loss distribution for the pool of reference instruments.

Going one step further and making the additional simplifying assumption of a large homogeneous portfolio (LHP), i.e. assuming it is possible to approximate the real reference credit portfolio with a portfolio consisting of a large number of equally weighted identical instruments (having the same term structure of default probabilities, recovery rates, and correlations to the common factor), we get a closed-form analytic synthetic CDO pricing formula. This LHP limit approximation employing the Law of Large Numbers was first proposed by Vasicek [14], [15].

In the case of the one factor Gaussian copula all integrals in the pricing formulas can be computed analytically (see [9]). Due to its simplicity, this model has become market standard. However, there is a fundamental problem: if we calculate the correlations that are implied by the market prices of tranches of the same CDO using the LHP approach, we do not get the same correlation over the whole structure but observe a correlation smile. The main explanation of this phenomenon is the lack of tail dependence of the Gaussian copula. Various authors have proposed different ways to bring more tail dependence into the model. One approach is the introduction of additional stochastic factors into the model. Andersen and Sidenius [1] extended the Gaussian factor copula model to random recovery and random factor loadings. Trinh et al [13] allowed for idiosyncratic and systematic jumps to default. Many other authors proposed a different approach - to use a copula that exhibits more tail dependence: Examples are the Marshall-Olkin copula in Andersen and Sidenius [1], the Student t copula in O'Kane and Schloegl [10], and the double t distribution in Hull and White [7].

Burtschell, Gregory and Laurent [5] performed a comparative analysis of a Gaussian copula model, stochastic correlation extension to Gaussian copula, Student t copula model, double t factor model, Clayton copula and Marshall-Olkin copula. By pricing the tranches of DJ iTraxx they showed that Student t and Clayton copula models provided results very similar to the Gaussian copula model. The Marshall-Olkin copula lead to a dramatic fattening of the tail. The results of the double t factor model and stochastic correlation model were closer to the market quotes, and the factor loading model of Andersen [1] performed similar to the latter ones.

Unfortunately, the integrals in the synthetic CDO pricing formula in the LHP model that is based on the double t copula cannot be computed analytically. The major problem is the instability of the Student t distribution under convolution. The calculation of the default thresholds (that are quantiles of the distribution of asset returns) requires a numerical root search procedure involving numerical integration that increases the computation time dramatically (see section 3). Thus, finding a different heavy tailed distribution that is similar to the Student t but stable under convolution would help to decrease the computation time tremendously. As computation time is an important issue for a large range of applications such as the determination of an optimal portfolio asset allocation (including CDO tranches) where CDO tranches have to be repriced in each scenario path at each time step in the future, the usage of such a distribution is crucial.

In our opinion, the Normal Inverse Gaussian (NIG) distribution is an appropriate distribution to solve the problem. The family of NIG distributions is a special case of the generalised hyperbolic distributions (Barndorff-Nielsen [3]). Due to their specific characteristics, NIG distributions are very interesting for applications in finance - they are a generally flexible four parameter distribution family that can produce fat tails and skewness, the class is convolution stable under certain conditions and the cumulative distribution function, density and inverse distribution functions can still be computed sufficiently fast ([12] and chapter four). The distribution has been employed, e.g., for stochastic volatility modelling (Barndorff-Nielsen [4]).

This paper is organised as follows: In the first section we recall the general pricing approach for synthetic CDOs. The one factor Gaussian copula model and the LHP approach are discussed in the second section. In the third section we present the extension of LHP to a double t factor model. The fourth section contains a brief review of the properties of the NIG distribution including a short description of its efficient implementation. In the fifth section we present the one factor NIG copula model and the CDO pricing formulas for our NIG LHP model. Finally, the last two sections provide empirical investigations and comparative analyses of the pricing abilities of the three models and their tail dependences.

1 General semi-analytic approach for pricing synthetic CDOs

We consider a synthetic CDO with a reference portfolio consisting of credit default swaps only. A protection seller of a synthetic CDO tranche receives from the protection buyer spread payments on the outstanding notional at regular payment dates (usually quarterly). If the total loss of the reference credit portfolio exceeds the notionals of the subordinated tranches, the protection seller has to make compensation payments for these losses to the protection buyer.

Basically, the pricing of a synthetic CDO tranche that takes losses from K_1 to K_2 (with $0 \leq K_1 < K_2 \leq 1$) of the reference portfolio works in the same way as the pricing of a credit default swap. Let's assume that

$$0 <= t_0 < ... < t_{n-1} \tag{1}$$

denote the spread payment dates, and T (with $t_{n-1} < t_n = T$) is the maturity of the synthetic CDO.

Then, the value of the premium leg of the tranche is the present value of all expected spread payments:

$$\text{Premium Leg} = \sum_{i=1}^{n} \Delta t_i \cdot spread \cdot \left(1 - EL_{(K_1,K_2)}\,(t_{i-1})\right) \cdot B(t_0, t_{i-1}), \tag{2}$$

where $\Delta t_i = t_i - t_{i-1}$, $B(t_0, t_i)$ is the discount factor and $EL_{(K_1,K_2)}\,(t_i)$ is the expected percentage loss of the $(K_1 - K_2)$ CDO tranche. The value of the protection leg can be calculated according to:

$$\begin{aligned} \text{Protection Leg} &= \int_{t_0}^{t_n} B(t_0, s) dEL_{(K_1,K_2)}\,(s) \\ &\approx \sum_{i=1}^{n} \left(EL_{(K_1,K_2)}\,(t_i) - EL_{(K_1,K_2)}\,(t_{i-1})\right) \cdot B(t_0, t_i). \end{aligned} \tag{3}$$

At issuance of the CDO tranche the tranche spread is determined so that the values of premium leg and protection leg are equal. Note, that all expectations are calculated under the risk neutral measure.

Assume, we had already computed the time t discrete loss distribution of the reference portfolio (for simplicity we omit the index t):

$$\{L_k \text{ with probability } p_k\}_{k=1,\ldots,m}.$$

Then the $(K_1 - K_2)$ CDO tranche suffers a loss of $(\min(L_k, K_2) - K_1)^+$ with probability p_k and the percentage expected loss of the tranche can be easily calculated:

$$EL_{(K_1,K_2)} = \frac{1}{K_2 - K_1} \sum_{k=1}^{m} (\min(L_k, K_2) - K_1)^+ \cdot p_k. \tag{4}$$

Given a continuous portfolio loss distribution function $F(x)$, the percentage expected loss of the $(K_1 - K_2)$ CDO tranche can be computed as:

$$EL_{(K_1,K_2)} = \frac{1}{K_2 - K_1} \left(\int_{K_1}^{1} (x - K_1) dF(x) - \int_{K_2}^{1} (x - K_2) dF(x) \right). \tag{5}$$

Thus, the central problem in the pricing of a CDO tranche is to derive the loss distribution of the reference portfolio.

In the next sections we present the factor copula model of correlated defaults as well as semi-analytical and analytical approximation methods to compute the portfolio loss distribution and the expected loss of a tranche.

2 One factor Gaussian model and LHP approximation

The LHP approach is based on a one factor Gaussian copula model of correlated defaults. Assume that the portfolio of reference assets consists of m financial instruments and the asset return until time t (further we omit t for simplicity of the notations) of the i-th issuer in the portfolio, A_i, is assumed to be of the form:

$$A_i = a_i M + \sqrt{1 - a_i^2} X_i, \tag{6}$$

where $M, X_i, i = 1, ..., m$, are independent standard normally distributed random variables. Then, conditionally on the common market factor M, the asset returns of the different issuers are independent. Note, that due to the stability of normal distributions under convolution the asset return A_i follows a standard normal distribution as well.

According to Merton's approach we assume that default occurs when the asset return of obligor i crosses the threshold C_i, which is implied by the obligor's default probability q_i :

$$q_i = P\left[A_i \leq C_i\right] = \Phi\left(C_i\right), \tag{7}$$

where Φ is the standard normal distribution function.

The model is calibrated to observable market prices of credit default swaps, i.e. the default thresholds are chosen so that they produce risk neutral default probabilities implied by quoted credit default swap spreads:

$$C_i = \Phi^{-1}(q_i).$$

According to equation (6), the i-th issuer defaults if

$$X_i \leq \frac{C_i - a_i M}{\sqrt{1 - a_i^2}}.$$

Then the probability that the i-th issuer defaults conditional on the factor M is

$$p_i(M) = \Phi\left(\frac{C_i - a_i M}{\sqrt{1 - a_i^2}}\right). \tag{8}$$

If we assume that the portfolio is homogeneous, i.e. $a_i = a$ and $C_i = C$ for all i and the notional amounts and recovery R are the same for all issuers, then the default probability of all issuers in the portfolio conditional on M is given by

$$p(M) = \Phi\left(\frac{C - aM}{\sqrt{1 - a^2}}\right). \tag{9}$$

and the loss distribution of the portfolio can be computed as follows – the probability of the percentage portfolio loss L being $L_k = \frac{k}{m}(1 - R)$ is equal to the probability that exactly k out of m issuers default:

$$P\left[L = L_k | M\right] = \begin{pmatrix} m \\ k \end{pmatrix} \Phi\left(\frac{C - aM}{\sqrt{1 - a^2}}\right)^k \left(1 - \Phi\left(\frac{C - aM}{\sqrt{1 - a^2}}\right)\right)^{m-k}. \tag{10}$$

Due to conditional independence and only two possible states the conditional loss distribution is binomial. The unconditional loss distribution $P[L = L_k]$ can be obtained by integrating equation (10) with the distribution of the factor M:

$$P[L = L_k] = \binom{m}{k} \int_{-\infty}^{\infty} \Phi\left(\frac{C - au}{\sqrt{1-a^2}}\right)^k \left(1 - \Phi\left(\frac{C - au}{\sqrt{1-a^2}}\right)\right)^{m-k} d\Phi(u). \qquad (11)$$

Since the calculation of the loss distribution in (10) is quite computationally intensive for large m, it is desirable to use some approximation. The large portfolio limit approximation proposed by Vasicek [15], [14] is a very simple but powerful method.

For simplicity let us first assume a zero recovery rate.

We consider the cumulative probability of the percentage portfolio loss not exceeding $\theta \in [0, 1]$

$$F_m(\theta) = \sum_{k=0}^{[m\theta]} P[L = L_k].$$

Substituting $s = \Phi\left(\frac{C-au}{\sqrt{1-a^2}}\right)$ and plugging in equation (11) we get the following expression for $F_m(\theta)$:

$$F_m(\theta) = \sum_{k=0}^{[m\theta]} \binom{m}{k} \int_0^1 s^k (1-s)^{m-k} d\Phi\left(\frac{\sqrt{1-a^2}\Phi^{-1}(s) - C}{a}\right). \qquad (12)$$

Since

$$\lim_{m\to\infty} \sum_{k=0}^{[m\theta]} \binom{m}{k} s^k (1-s)^{m-k} = \begin{cases} 0, & \text{if } \theta < s \\ 1, & \text{if } \theta > s \end{cases}$$

the cumulative distribution of losses of a large portfolio equals

$$F_\infty(\theta) = \Phi\left(\frac{\sqrt{1-a^2}\Phi^{-1}(\theta) - C}{a}\right). \qquad (13)$$

Therefore, in case of the large homogeneous portfolio assumption it is possible to compute the integrals in (5) analytically and the expected loss of the tranche is given by:

$$EL_{(K_1,K_2)} = \frac{\Phi_2\left(-\Phi^{-1}(K_1), C, -\sqrt{1-a^2}\right) - \Phi_2\left(-\Phi^{-1}(K_2), C, -\sqrt{1-a^2}\right)}{K_2 - K_1},$$

where Φ_2 is the bivariate normal distribution function.

Now, let us assume that assets have the same (maybe non-zero) recovery rate R. Then, the total loss of the equity tranche of K will occur only when assets of the total amount of $\frac{K}{1-R}$ have defaulted. Thus, the expected loss of the tranche between K and 1 is given by

$$\begin{aligned} EL^R_{(K,1)} &= \int_{\frac{K}{1-R}}^{1} (1-R)\left(x - \frac{K}{1-R}\right) dF_\infty(x) \\ &= (1-R)\, EL_{\left(\frac{K}{1-R},1\right)}, \end{aligned}$$

where EL^R denotes the loss under the recovery rate R. Finally, it is easy to see that the expected percentage loss of the mezzanine tranche taking losses from K_1 to K_2 under the assumption of a constant recovery rate R is

$$EL^R_{(K_1,K_2)} = EL_{\left(\frac{K_1}{1-R},\frac{K_2}{1-R}\right)}. \tag{14}$$

3 LHP with double t copula

One natural extension of the LHP approach is to use a distributional assumption that produces heavy tail. The double t one factor model proposed by Hull and White [7] assumes Student t distributions for the common market factor M as well as for the individual factors X_i. Then the loss distribution F_∞ in (13) becomes:

$$F_\infty(\theta) = T\left(\frac{\sqrt{1-a^2}T^{-1}(\theta) - C}{a}\right),$$

where T denotes the Student t distribution function. Unfortunately, it is not possible to solve the integral in (5) analytically using this loss distribution and one has to use some numerical integration method.

The asset returns A_i do not follow necessarily Student t distributions since the Student t distribution is not stable under convolution. The distribution function H_i of A_i must be computed numerically. Afterwards, it is possible to calculate the default thresholds C_i by $H_i^{-1}(q_i)$. This procedure is quite time consuming and it makes the double t model too slow for Monte Carlo based risk management applications.

4 NIG distribution and its main properties

4.1 Definition and properties of the NIG distribution

The normal inverse Gaussian distribution is a mixture of normal and inverse Gaussian distributions.

A non-negative random variable Y has inverse Gaussian distribution with parameters $\alpha > 0$ and $\beta > 0$ if its density function is of the form:

$$f_{\mathcal{IG}}(y;\, \alpha,\, \beta) = \begin{cases} \dfrac{\alpha}{\sqrt{2\pi\beta}}\, y^{-3/2} \exp\left(-\dfrac{(\alpha-\beta y)^2}{2\beta y}\right) & ,\ \text{if } y > 0 \\ 0 & ,\ \text{if } y \leq 0. \end{cases}$$

A random variable X follows a Normal Inverse Gaussian (NIG) distribution with parameters α, β, μ and δ if:

$$\begin{aligned} X \mid Y = y &\sim \mathcal{N}(\mu + \beta y,\, y) \\ Y &\sim \mathcal{IG}(\delta\gamma,\, \gamma^2) \text{ with } \gamma := \sqrt{\alpha^2 - \beta^2}, \end{aligned}$$

with parameters satisfying the following conditions: $0 \leq |\beta| < \alpha$ and $\delta > 0$. We then write $X \sim \mathcal{NIG}(\alpha,\, \beta,\, \mu,\, \delta)$ and denote the density and probability functions by $f_{\mathcal{NIG}}(x;\, \alpha,\, \beta,\, \mu,\, \delta)$ and $F_{\mathcal{NIG}}(x;\, \alpha,\, \beta,\, \mu,\, \delta)$ correspondingly.

The density of a random variable $X \sim \mathcal{NIG}(\alpha,\, \beta,\, \mu,\, \delta)$ is given by:

$$f_{\mathcal{NIG}}(x;\, \alpha,\, \beta,\, \mu,\, \delta) = \frac{\delta\alpha \cdot \exp\left(\delta\gamma + \beta(x-\mu)\right)}{\pi \cdot \sqrt{\delta^2 + (x-\mu)^2}} K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right),$$

where $K_1(w) := \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}w(t + t^{-1})\right) dt$ is the modified Bessel function of the third kind. The NIG moment generating function $M(t) = E[\exp(tx)]$ is given by:

$$M_{\mathcal{NIG}}(x;\, \alpha,\, \beta,\, \mu,\, \delta) = \exp(\mu t)\, \frac{\exp\left(\delta\sqrt{\alpha^2 - \beta^2}\right)}{\exp\left(\delta\sqrt{\alpha^2 - (\beta + t)^2}\right)}.$$

The main properties of the NIG distribution class are the scaling property

$$X \sim \mathcal{NIG}(\alpha,\, \beta,\, \mu,\, \delta) \Rightarrow cX \sim \mathcal{NIG}\left(\frac{\alpha}{c},\, \frac{\beta}{c},\, c\mu,\, c\delta\right), \tag{15}$$

and the closure under convolution for independent random variables X and Y

$$\begin{aligned} &X \sim \mathcal{NIG}(\alpha,\, \beta,\, \mu_1,\, \delta_1), Y \sim \mathcal{NIG}(\alpha,\, \beta,\, \mu_2,\, \delta_2) \\ &\quad \Rightarrow X + Y \sim \mathcal{NIG}\left(\alpha,\, \beta,\, \mu_1 + \mu_2,\, \delta_1 + \delta_2\right). \end{aligned} \tag{16}$$

The moments (mean, variance, skewness and kurtosis) of a random variable $X \sim \mathcal{NIG}(\alpha,\, \beta,\, \mu,\, \delta)$ are:

$$\begin{aligned} \mathbb{E}(X) &= \mu + \delta\frac{\beta}{\gamma} & \mathbb{V}(X) &= \delta\frac{\alpha^2}{\gamma^3} \\ \mathbb{S}(X) &= 3\frac{\beta}{\alpha \cdot \sqrt{\delta\gamma}} & \mathbb{K}(X) &= 3 + 3\left(1 + 4\left(\frac{\beta}{\alpha}\right)^2\right)\frac{1}{\delta\gamma}. \end{aligned}$$

4.2 Efficient implementation of the NIG distribution

So far we have seen that, theoretically, it is much more convenient to work with the NIG distribution than with the Student t distribution. Still, it is not obvious if the NIG distribution, especially due to its rather complicated density function involving a Bessel function, can be implemented efficiently. As the main driver of our considerations has been speed of computation, an efficient implementation is crucial. In the following paragraphs we briefly describe our implementation in Matlab.

- The implementation of the NIG distribution can be done straightforward using the pretty efficient Matlab version of the Bessel function.

- Instead of only using the obvious representation of

  $$F_{\mathcal{NIG}}(x) = \int_{-\infty}^{x} f_{\mathcal{NIG}}(t)dt$$

  we additionally use the fact that the NIG distribution stems from a convolution of the normal and the inverse Gaussian distribution:

  $$F_{\mathcal{NIG}}(x) = \int_{0}^{\infty} \Phi\Big(\frac{x-(\mu+\beta y)}{\sqrt{y}}\Big) f_{\mathcal{IG}}(y;\delta\gamma,\gamma^2)dy$$

  Depending on the number of points where the same distribution function is evaluated, we either use the first or the second representation. If only evaluation of a few points is necessary, the second version is more efficient. If more points have to be calculated, the first representation can be used and the integral is split into appropriate small integrals.

  Both integrals do not live on a compact interval, hence we further split up the integral in a compact part and the rest. For this remainder, we use a logarithmic transformation. If we use for example $t = \exp(-y)$ the latter integral is transformed to the compact interval

  $$F_{\mathcal{NIG}}(x) = \int_{0}^{1} \Phi\Big(\frac{x-(\mu-\beta\log(t))}{\sqrt{\log(t)}}\Big) f_{\mathcal{IG}}(-\log(t),\delta\gamma,\gamma^2)\frac{1}{t}dt$$

  We approximate all occuring integrals by a Gauss cubature formula using four Gauss nodes. Depending on the required accuracy we recursively integrate on subdivided integration domains.

- For the calculation of the inverse distribution function – even if it is only evaluated at one point y – we table the distribution function $F_{\mathcal{NIG}}$ on a fine grid around the mean (with size equal to several standard deviations). This table is then used to find an x with $F_{\mathcal{NIG}}(x) = y$ by using linear interpolation inside the grid and logarithmic interpolation outside the grid. This can still be done much faster than Newton's method for root finding for $F_{\mathcal{NIG}}$.

As we can see from table 1 the computation time is a factor 6 faster than for the Student t example, where we have used similar tricks to speed-up of the computation, while working with a much more flexible distribution.

5 LHP with NIG copula

We define the one factor NIG copula model in the following way. The asset return of the i-th issuer in a portfolio A_i is assumed to be of the form:

$$A_i = aM + \sqrt{1-a^2}X_i,$$

where $M, X_i, i = 1, ..., m$ are independent NIG random variables with the following parameters:

$$\begin{aligned}
M(t) &\sim \mathcal{NIG}\left(\alpha,\, \beta,\, -\frac{\beta\gamma^2}{\alpha^2},\, \frac{\gamma^3}{\alpha^2}\right) \\
X_i(t) &\sim \mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\alpha,\, \frac{\sqrt{1-a^2}}{a}\beta,\, -\frac{\sqrt{1-a^2}}{a}\frac{\beta\gamma^2}{\alpha^2},\, \frac{\sqrt{1-a^2}}{a}\frac{\gamma^3}{\alpha^2}\right),
\end{aligned}$$

with $\gamma = \sqrt{\alpha^2 - \beta^2}$.

Then, using the scaling property and stability under convolution of NIG distribution in (15) and (16) we get NIG distributed asset returns:

$$A_i(t) \sim \mathcal{NIG}\left(\frac{\alpha}{a},\, \frac{\beta}{a},\, -\frac{1}{a}\frac{\beta\gamma^2}{\alpha^2},\, \frac{1}{a}\frac{\gamma^3}{\alpha^2}\right).$$

The third and the fourth parameters are chosen such that we get expected value of zero and variance of one. For non-zero β we get a skewed distribution. To simplify notations we denote $F_{\mathcal{NIG}}\left(x;\, s\alpha,\, s\beta,\, -s\frac{\beta\gamma^2}{\alpha^2},\, s\frac{\gamma^3}{\alpha^2}\right)$ by $F_{\mathcal{NIG}(s)}(x)$.
One big advantage of the NIG copula is that the default thresholds are easy and fast to compute:

$$C = F^{-1}_{\mathcal{NIG}(\frac{1}{a})}(p).$$

The default probability conditional on M of each instrument in the portfolio is given by

$$p(M) = F_{\mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\right)}\left(\frac{C - aM}{\sqrt{1-a^2}}\right),$$

and the loss distribution function of the large homogeneous portfolio is

$$F_\infty(x) = 1 - F_{\mathcal{NIG}(1)}\left(\frac{C - \sqrt{1-a^2}F^{-1}_{\mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\right)}(x)}{a}\right).$$

Table 1: Pricing DJ iTraxx tranches with the LHP model based on different distributions

	Market	Gaussian	t(4)-t(4)	t(3)- t(3)	NIG(1)	NIG(2)
0-3%	23,53%	23,53%	23,53%	23,53%	23,53%	23,53%
3-6%	62,75 bp	140,46 bp	73,3 bp	53,88 bp	62,75 bp	62,75 bp
6-9%	18 bp	29,91 bp	28,01 bp	23,94 bp	27,9 bp	27,76 bp
9-12%	9,25 bp	7,41 bp	16,53 bp	15,96 bp	17,64 bp	17,42 bp
12-22%	3,75 bp	0,8 bp	8,68 bp	9,94 bp	9,79 bp	9,6 bp
absolute error		94,41 bp	32,82 bp	27,82 bp	24,34 bp	23,77 bp
correlation		15,72%	19,83%	18,81%	16,21%	15,94%
α					0,4794	0,6020
β					0	-0,1605
comp. time		0,5 s	12,6 s	11 s	1,5 s	1,6 s

6 Comparison of the models: pricing DJ iTraxx

In order to compare the properties of the LHP model with Gaussian, double t and NIG copulas, we consider the example of the tranched Dow Jones iTraxx Europe with 5 years maturity. The reference portfolio consists of 125 credit default swap names. The standard tranches have attachment/detachment points at 3%, 6%, 9%, 12% and 22%. The investors of the tranches receive quarterly spread payments on the outstanding notional and compensate for losses when these hit the tranche they are invested in. The investor of the equity tranche receives an up-front fee that is quoted in the market and an annual spread of 500 bp quarterly. The settlement date of the fourth series of this index is 20-March-2006 and matures on 20-June-2011. We consider the market quotes of iTraxx tranches at 12-April-2006. The average CDS spread of the corresponding CDS portfolio is 32 bp at this day. We use the constant default intensity model to derive the marginal default distributions, and estimate the default intensity of the large homogeneous portfolio from the average portfolio CDS spread. The constant recovery rate is assumed to be 40%.

Gaussian and double t factor copulas have only one parameter, the correlation. We estimate this parameter so that the price of the equity tranche fits the market quote, i.e. we calculate the implied equity correlation. The same correlation is used to price the other tranches. The versions of the NIG factor copula we consider have one parameter α ($\beta = 0$) or two parameters α and β besides the correlation. We use least squares estimation to estimate these parameters such that they fit all observed tranche spreads.

Table 1 presents the market quotes of the iTraxx tranches as well as the prices of the LHP model with Gaussian one factor copula, double t distribution with 3 and 4 degrees of freedom and NIG factor copula with one and two free parameters. In the one parameter NIG copula, the parameter β is set to zero which makes the distribution symmetric. The double t factor copula models fit only the equity tranche exactly since it has only one continuous valued parameter (correlation). The double t model with 3 degrees of freedom underprices the second tranche while the double t model with 4 degrees of freedom over-prices it. Since the second model parameter (degrees of freedom) is only integer valued it is in general impossible to fit the second tranche exactly. The results of the NIG copulas are similar to the results of double t copulas. The additional free parameter in the NIG copula makes it more flexible: the second tranche can be fitted exactly as well. Surprising is that one more free parameter β doesn't bring much improvement to the fitting results in this example.

We plot density and cumulative distribution functions of portfolio losses from the LHP model with Gaussian, double t(3) and NIG factor copulas in Figure 1. Figure 2 shows the differences between modified (t or NIG) and Gaussian densities. It is especially easy to see that all modified models redistribute risk out of the lower end of the equity tranche to its higher end. Note that the total risk difference within the equity tranche is zero since we have calibrated all models to fit the equity tranche. The Gaussian model allocates more risk to mezzanine tranches than the modified models. Figure 3 shows that NIG(2) copula allocates slightly more risk at the 3-6% tranche than the double t(3) copula does. Figure 4 presents the density function of the asset returns. The density of the NIG(2) model is slightly skewed to the right.

We have reformulated the prices produced by all models into base correlations (Figure 5). Base correlations are the implied Gaussian model correlations of the corresponding equity tranches, i.e. of 0-3%, 0-6%, 0-9%, 0-12% and 0-22% tranches. The NIG model with two parameters fits the first two market base correlations exactly.

Base correlation is very popular among market participants since due to its monotonicity it can be easily interpolated and used to price the off-market tranches with the Gaussian model. However, the big problem of this method is that it can produce arbitrage prices. The big advantage of our NIG model is that we can compute the price of any off-market tranche with the same correlation value that guarantees arbitrage-free pricing.

7 Comparison of the tail dependence

In this section we investigate the tail dependence of the one factor copulas under study. In particular, the amount of dependence in the lower-left-quadrant is relevant for modelling credit portfolios.

Let X_1 and X_2 be continuous random variables. We consider the coefficient of lower tail dependence:

$$\lambda_L(x) = P\{X_2 \leq x | X_1 \leq x\}.$$

Random variables X_1 and X_2 are said to be asymptotically dependent in the lower tail if $\lambda_L > 0$ and asymptotically independent in the lower tail if $\lambda_L = 0$, where

$$\lambda_L = \lim_{x \to -\infty} \lambda_L(x).$$

The coefficient of upper tail dependence is defined as

$$\lambda_U(x) = P\{X_2 > x | X_1 > x\},$$

and X_1 and X_2 are asymptotically dependent in the upper tail if $\lambda_U > 0$, where

$$\lambda_U = \lim_{x \to \infty} \lambda_U(x).$$

We plot the tail dependence coefficients of Gaussian, double t(5) and NIG factor copulas in Figure 6. The Gaussian copula shows no tail dependence. The tail dependence coefficient of double t(3) copula is significantly larger than that of the Gaussian copula. The tail dependence of the NIG copula with one parameter is slightly lower than the tail dependence of the double t(3) copula. Note, that the upper and the lower tail dependence structures of Gaussian, double t(3) and NIG(1) copulas are symmetric. This is not the case for the two parameter NIG copula. This copula has a higher lower tail dependence coefficient similar to that of the NIG(1) copula, and a very small upper tail dependence coefficient.

These properties of the copulas under study can be observed in Figure 7 as well. We have simulated pairs of correlated asset returns with Gaussian, double t with 3 degrees of freedom, NIG with one and two parameters factor copulas and plotted the contours of their joint densities. The double t factor copula produces more extreme values than the other copulas. The NIG copula with two parameters has more extreme values in the lower left tail.

Conclusion

This paper has introduced the NIG factor copula as an extension of the LHP model for pricing synthetic CDOs. We have presented an analytical formula for the distribution function of the portfolio loss. In general, the NIG distribution has four free parameters. The standardised symmetric NIG distribution with zero mean and unit standard deviation still has one free real parameter. The ability of this structure to fit CDO tranches is similar to that of the double t copula that has one free integer parameter. However, NIG can fit the second tranche exactly what double t in general cannot do. Furthermore, we have studied the properties of the one factor copula model with a skewed NIG distribution having two free parameters. This copula has non symmetric upper and lower tail dependence. The two parameter NIG factor copula model could bring only a very slight improvement in our example.

The main purpose of developing the NIG factor copula was simplification and speeding up computation of the default thresholds. Incorporating the effect of tail dependence into this simple one factor credit portfolio model by means of replacing Gaussian distribution with Student t distribution for the factors has the following disadvantage: it is not possible to compute the distribution function of asset returns analytically because of the lack of stability under convolution. This leads to a dramatic increase of computation time and makes it impossible to use this model for some important applications in practice. The NIG distribution still has a higher tail dependence than the Gaussian distribution. In addition, it is stable under convolution under certain conditions. The employment of the NIG distribution does not only bring significant improvement with respect to computation times but also more flexibility in the modelling of the dependence structure of a credit portfolio.

References

[1] Leif Andersen and Jakob Sidenius. Extensions to the Gaussian copula: random recovery and random factor loadings. *Journal of credit risk*, 1(1), 2005.

[2] Leif Andersen, Jakob Sidenius, and Susanta Basu. All your hedges in one basket. *Risk*, November 2003.

[3] O.E. Barndorff-Nielsen. Hyperbolic distributions and distributions on hyperbolae. *Scandinavian Journal of statistics*, 5:151–157, 1978.

[4] O.E. Barndorff-Nielsen. Normal Inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of statistics*, 24:1–13, 1997.

[5] X. Burtschell, J. Gregory, and L.-P. Laurent. A comparative analysis of CDO pricing models. Working paper, April 2005.

[6] P. Embrechts, F. Lindskog, and A. McNeil. *Modelling dependence with copulas and applications to risk management*, 2001.

[7] John Hull and Alan White. Valuation of a CDO and an n-th to default CDS without a Monte Carlo simulation. *Journal of Derivatives*, 2004.

[8] Lean-Paul Laurent and Jon Gregory. Basket default swaps, CDO's and factor copulas. Working paper, September 2003.

[9] D. O'Kane and L. Schloegl. Modelling credit: theory and practice. *Lehman Brothers Analytical Research Series*, February 2001.

[10] Dominic O'Kane and Lutz Schloegl. An analytical portfolio credit model with tail dependence. *Quantitative credit research*, January 2003. Lehman Brothers.

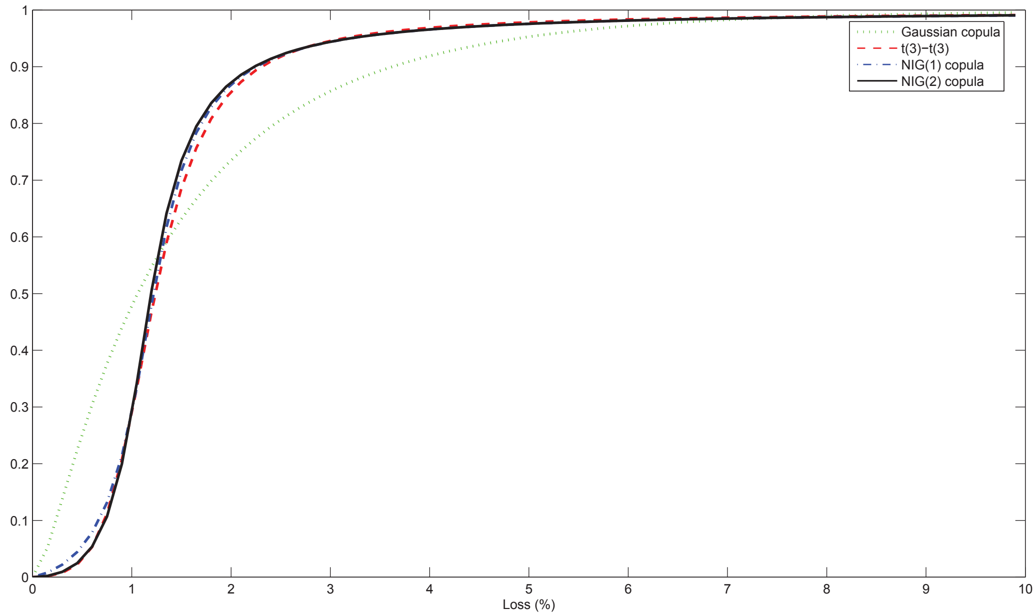
[11] Lutz Schloegl and Andrei Greenberg. The shape of implied loss distribution. *Quantitative credit research*, March 2005. Lehman Brothers.

[12] Michaela Tempes. Implementierung einer Copula-Toolbox unter Matlab. Master's thesis, TU Munich, 2005.

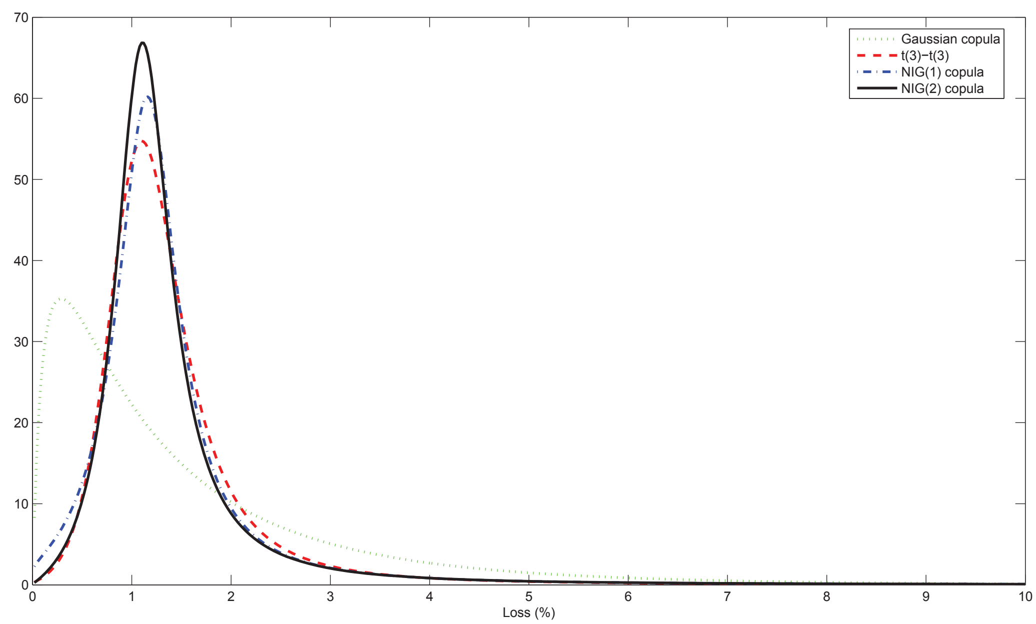
[13] Minh Trinh, Robin Thompson, and Mukundan Devarajan. Relative value of CDO tranches: a view through ASTERION. *Quantitative credit research*, March 2005. Lehman Brothers.

[14] Oldrich Vasicek. Limiting loan loss probability distribution. Memo, KMV corporation, August 1987.

[15] Oldrich Vasicek. Probability of loss on loan portflio. Memo, KMV corporation, February 1987.



(a) Cumulative distribution function



(b) Density function

Figure 1: Portfolio loss distribution from LHP model

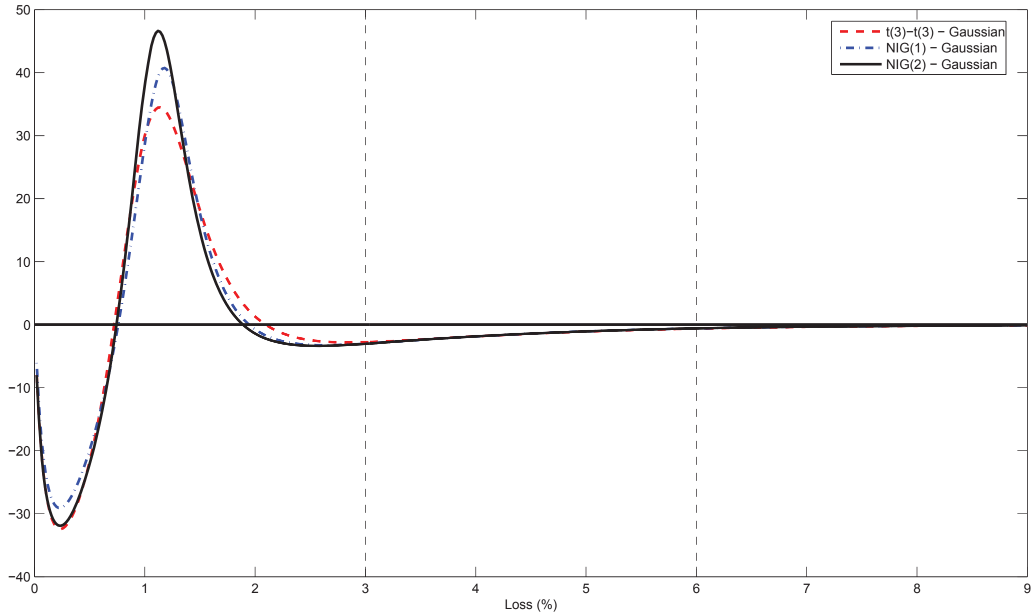


Figure 2: Difference between modified and Gaussian loss density

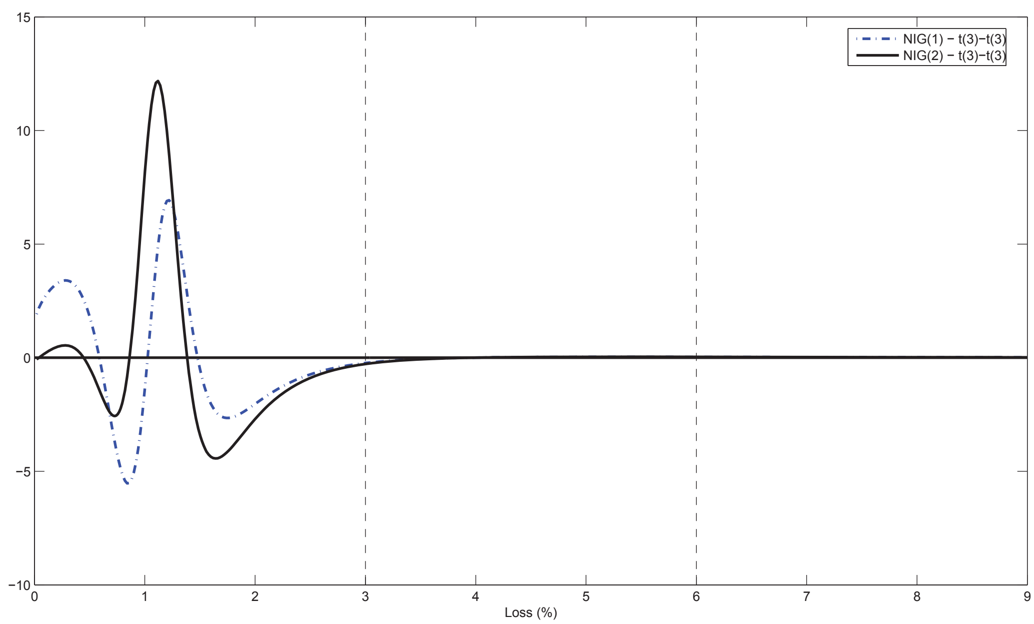


Figure 3: Difference between double t and NIG loss density

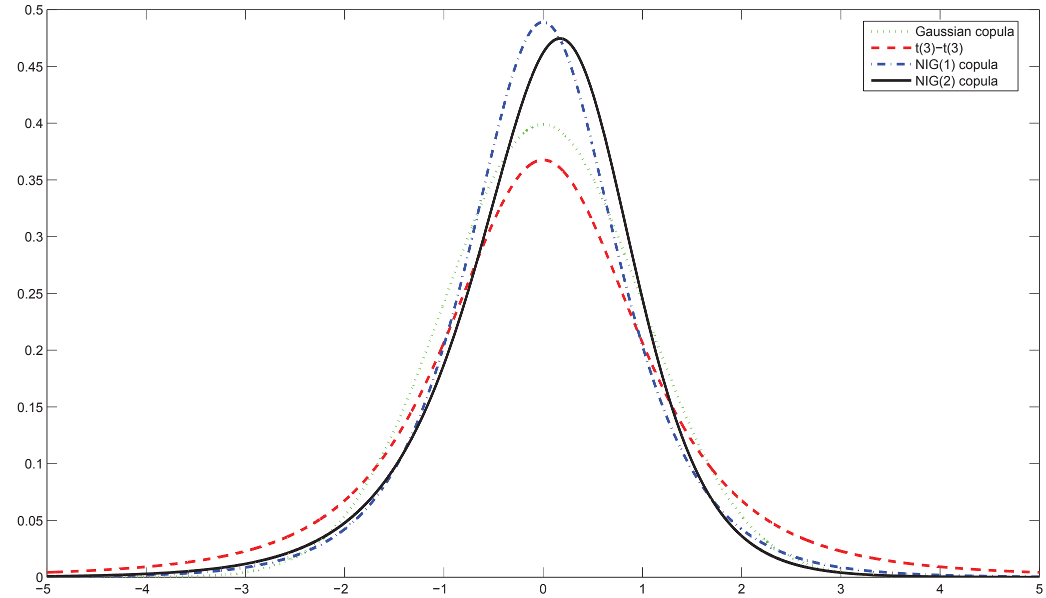


Figure 4: Distribution of asset returns

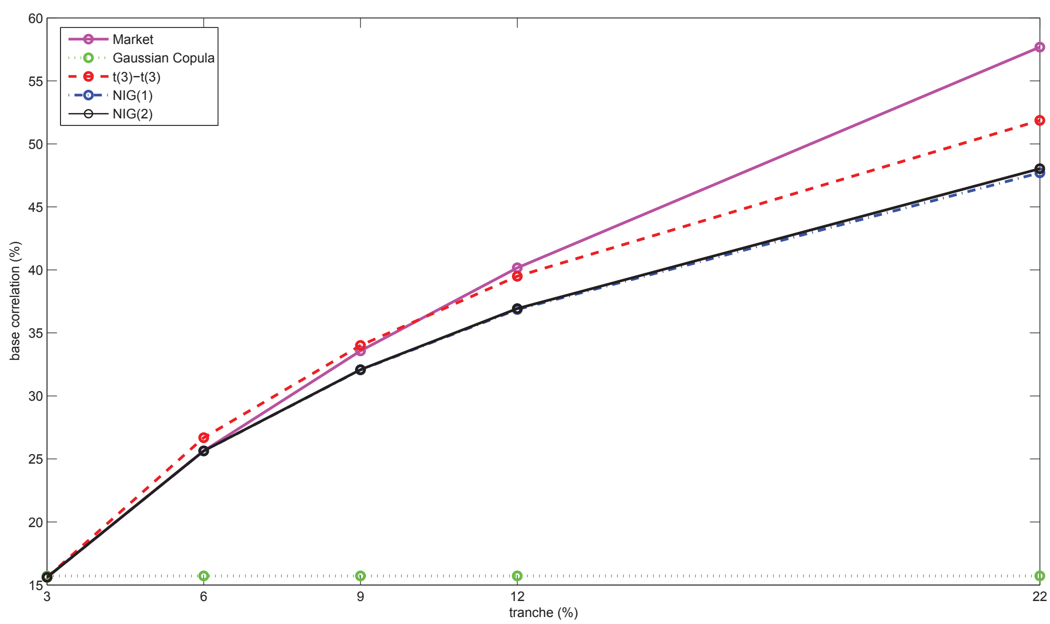
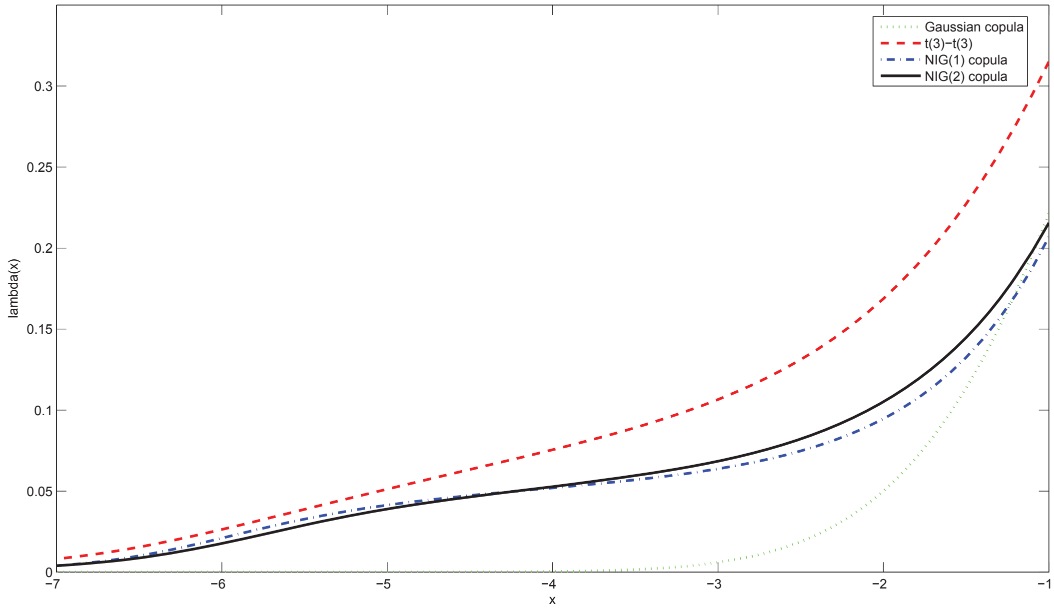
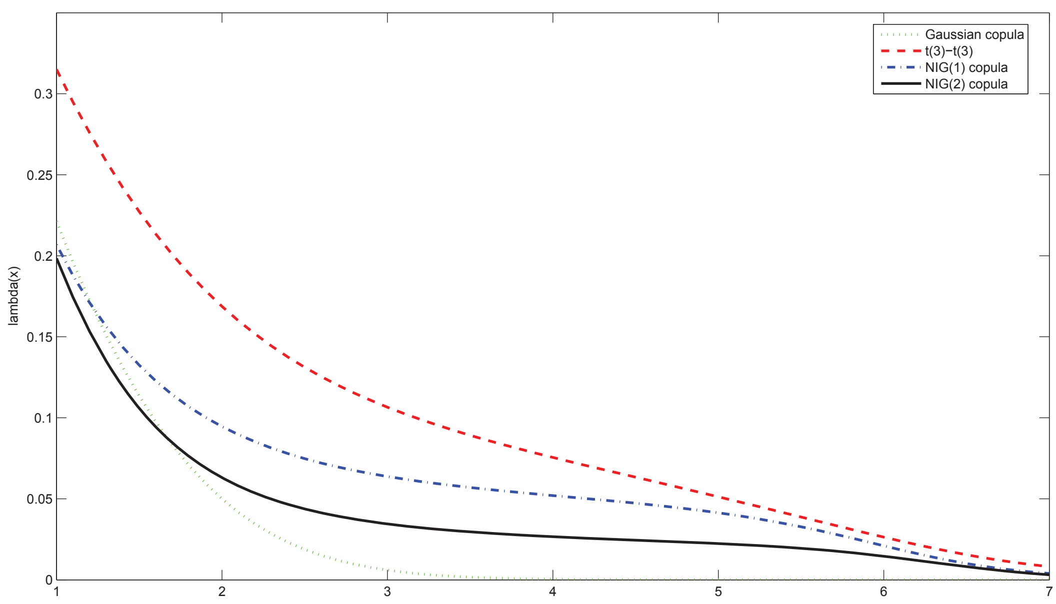


Figure 5: Implied base correlation



(a) Lower tail dependence



(b) Upper tail dependence

Figure 6: Tail dependence coefficients

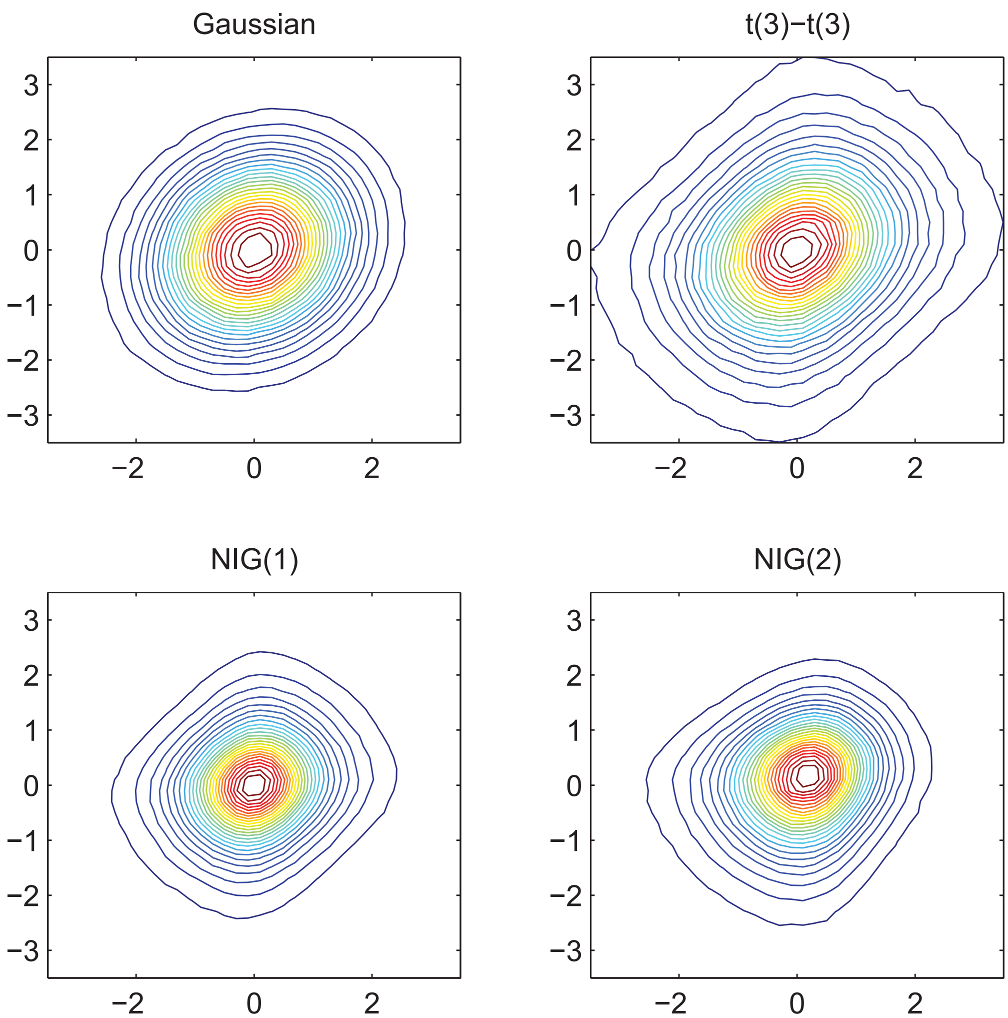


Figure 7: Density contours of one factor copula