Using Scenario Analysis for Risk Management

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Scenario analysis is an essential tool for financial risk management and asset allocation. It can give an important a priori information to a risk or portfolio manager and helps to control the effect of price changes to a portfolio, especially those of potential market crashes. This article deals with the problem of providing a set of consistent and reliable scenarios which may also consider market forecasts given by a research department or other market specialists. Bond and share prices are described by a multi-factor stochastic market model. The model is fitted and statistically examined using empirical data of the German bond market represented by the yields of the German Pfandbrief index (PEX) with maturities from one to ten years and the stock markets represented by the Dow Jones European stock market index (DJ Euro Stoxx 50) as well as the MSCI World Excluding EMU index (MSCI World ex EMU). A case study shows how this concept can be applied to risk management problems.

JEL classification index: C13, C32, E43, E47, G11

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1 Introduction

One of the most challenging problems in managing the risk of a portfolio or trading book is to be adequately prepared to potential future market changes. History may be one advisor for the future but definitely not the only and sometimes also not the best one. Nevertheless, empirical price changes can give an important insight into the joint behaviour of different risk factors under normal market conditions. Stochastic models may then be applied to describe the movement of future market prices and used to generate a corresponding set of price scenarios. Event or crash scenarios could be added to consider non-normal or chaotic market movements. However, if these scenarios are a good representation of the possible future price changes, the risk manager or trader may use this information to calculate risk numbers like expected return, standard deviation or shortfall probability, i.e. the probability that the return falls below a given benchmark. According to these numbers a portfolio manager may decide on the detailed structure of his portfolio. In this article we will use a multi-factor financial market model to describe the joint movement of bond and share prices. The model is described and theoretically solved in section 2. The bond market we consider is represented by the German Pfandbrief index (PEX) with maturities from one to ten years as it is published by the data provider Bloomberg on behalf of the Association of German Mortgage Banks and the Association of Public Banks. The stock market is described using the Dow Jones Euro Stoxx 50 (DJ Euro Stoxx 50) index which is a capitalization-weighted index of fifty European blue-chip stocks from those countries participating in the European and Monetary Union (EMU) and the MSCI World Excluding EMU (MSCI

World ex EMU) index which is a capitalization-weighted index monitoring the performance of stocks from around the world, excluding the countries that make up the EMU. In section 3 we use empirical price information to fit the financial market model to historical market prices and correlations. Based on specific (partial) market forecasts we develop a method which allows for the generation of a complete set of market scenarios consistent to these forecasts in section 4. In section 5 we use the generated scenario sets to calculate risk numbers according to unconditional or forecasted market movements. The results are applied to decide on the specific structure or structural changes of a portfolio or trading book.

2 The Model

In this section we develop and solve a multi-factor model to describe the joint evolution of share and bond prices. We will use a stock market index to model the evolution of the corresponding stock market. Individual share price behaviour may then be derived using the well-known capital asset pricing model or related methods (see, e.g. Elton and Gruber (1991) for more details). While the famous model of Black and Scholes (1973) is most widely accepted to be the standard model for describing stock price behaviour there are quite a number of stochastic models to describe a bond market. Usually these models use the short rate or the forward short rates, i.e. interest rates of an infinitesimal time to maturity measured at time t for time t (short rate) or for a future point in time (forward short rate). Among the most famous short rate models are those of Vasicek (1977), Cox et al. (1985), or Hull and White (1990). One of the most general frameworks is the forward short rate model introduced by Heath et al. (1992). In recent years a new class of market models was introduced by Brace et al. (1997) and Miltersen et al. (1997), the so-called LIBOR market models, as well as by Jamshidian (1998), the so-called swap market model. They describe the behaviour of market rates rather than that of the short or forward short rate. For an overview on interest rate models see, e.g. Hull (2000), Musiela and Rutkowski (1997), or Zagst (2002). In this paper we will use a Vasicek process to describe the PEX rates of different maturities which are published by Bloomberg each day. In this sense the model can be considered to be a new representative of the class of market models.

Following the famous model of Black and Scholes (1973), we describe the behaviour of a stock market index S_k, $k \in \{1, ..., N_S\}$, by the stochastic differential equation

$$dS_k(t) = S_k(t) \cdot (\mu_k dt + \sigma_k d\widetilde{W}_k(t)),\ t \in [0, T]\,,$$

with $\mu_k \in I\!R$ denoting the drift rate, $\sigma_k > 0$ the volatility of the stock market index, and with correlated one-dimensional Wiener processes $\widetilde{W}_k$. Using Itô's lemma (see, e.g., Itô (1951)) we can easily conclude that the corresponding process $(\ln (S_k (t)))_{t\in[0,T]}$ follows the stochastic differential equation

$$d \ln (S_k (t)) = \theta_k dt + \sigma_k d\widetilde{W}_k (t)$$

with $\theta_k := \mu_k - \frac{1}{2} \cdot \sigma_k^2$. Typical for this model is an exponential deterministic growth of the stock index plus random noise (see figure 1 for an empirical example).

On the other hand, interest rates or yields can usually be observed to drift around a long-term mean or mean reversion level due to economic cycles (see figure 2 for

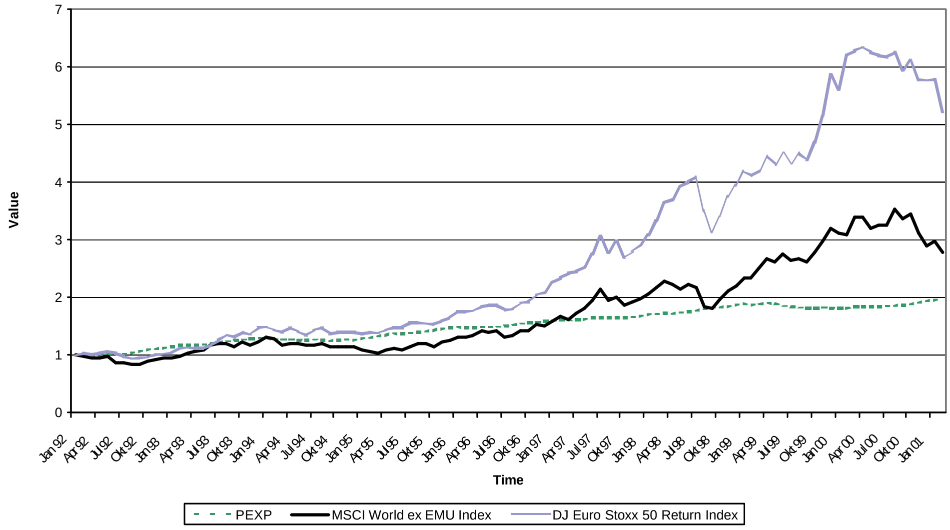

Figure 1: History of the PEXP, the performance index corresponding to the PEX, the DJ Euro Stoxx 50, and the MSCI World ex EMU with values normed to 1 at January 1992

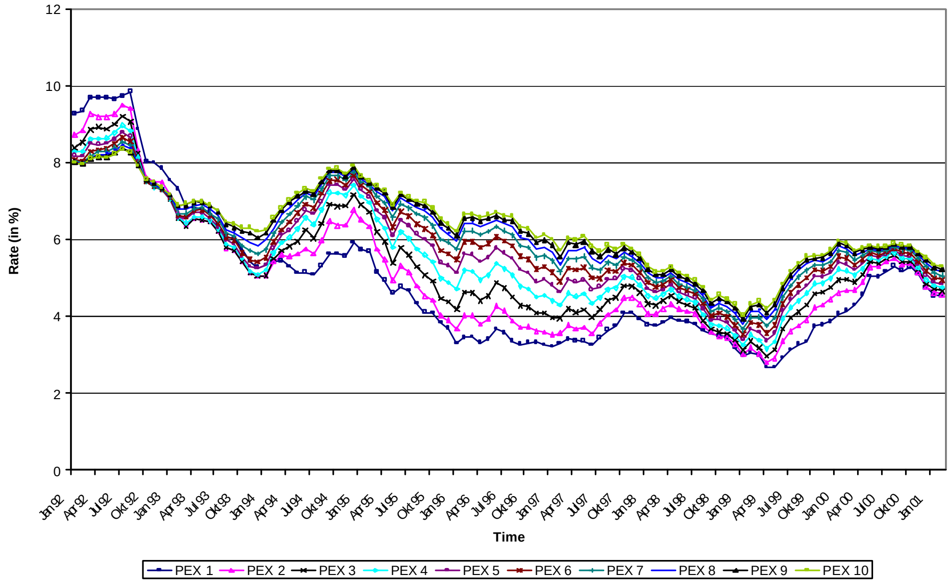

Figure 2: History of the PEX rates for maturities from 1 to 10 years

an empirical example). Using the model of Vasicek (1977) this behaviour can be described by the stochastic differential equation

$$dR_i\left(t\right)=\left(\theta_i-a_i\cdot R_i\left(t\right)\right)dt+\sigma_i d\widetilde{W}_i(t),\ t\in\left[0,T\right],\ i\in\left\{1,...,N_R\right\}$$

with R_i denoting the PEX yield-to-maturity (PEX rate) for a fixed time to maturity T_i, $a_i>0$ the mean reversion rate, $\sigma_i>0$ the volatility, and with correlated one-dimensional Wiener processes $\widetilde{W}_i$. Furthermore, $\frac{\theta_i}{a_i}$ with $\theta_i\geq 0$ is the mean reversion level of R_i, $i\in\{1,...,N_R\}$.

Using the previous models we describe a universe of different yields-to-maturity $(Y_i\left(t\right):=R_i\left(t\right),\ i=1,...,N_R)$ as well as the logarithm of different stock market indices $(Y_i\left(t\right):=\ln\left(S_i\left(t\right)\right),\ i=N_R+1,...,N:=N_R+N_S)$ by the stochastic differential equations

$$dY_i\left(t\right)=\left(\theta_i-a_i\cdot Y_i\left(t\right)\right)dt+\sigma_i d\widetilde{W}_i(t),\ i=1,...,N,\ t\in\left[0,T\right], \tag{1}$$

where we assume that $a_i=0$ for all stock market indices and that the Wiener processes $\widetilde{W}_i$, $i=1,...,N$, are correlated at a constant correlation rate ρ_{ik} defined by

$$Cov\left[d\widetilde{W}_i(t),d\widetilde{W}_k(t)\right]=\rho_{ik}dt,\ i,k\in\left\{1,...,N\right\}.$$

To solve this $N-$dimensional continuous time model, we orthogonalize the Wiener processes $\widetilde{W}_1,...,\widetilde{W}_N$ using the independent Wiener processes $W_1,...,W_N$ with $W_1:=\widetilde{W}_1$ and

$$\sigma_i d\widetilde{W}_i(t)=\sum_{j=1}^{N}\sigma_{ij}dW_j(t),\ t\in\left[0,T\right],$$

with $\sigma_{ij}\geq 0$ for all $i,j\in\{1,...,N\}$ and $\sigma_{ij}=0$ if $i<j$, i.e.

$$dY_i(t)=(\theta_i-a_i\cdot Y_i(t))dt+\sum_{j=1}^{N}\sigma_{ij}dW_j(t),\ i\in\left\{1,...,N\right\}.$$

This can be written as

$$dY\left(t\right)=\left[\theta+AY\left(t\right)\right]dt+\sigma dW\left(t\right) \tag{2}$$

with

$$\theta=\left(\theta_1,...,\theta_N\right)',\ \sigma=\left(\sigma_{ij}\right)_{\substack{i=1,...,N\\ j=1,...,N}}$$

and

$$A=\begin{pmatrix} -a_1 & 0 & \cdot & \cdot & \cdot & 0\\ 0 & & \cdot & & & \cdot\\ \cdot & & & \cdot & & \cdot\\ \cdot & & & & \cdot & \cdot\\ \cdot & & & & & 0\\ 0 & & \cdot & \cdot & \cdot\ 0 & -a_N \end{pmatrix}.$$

We can then state the following theorem which is proved in the appendix.

Theorem 2.1 *The solution to the stochastic differential equation* (2) *is given by*

$$Y_i\left(t\right)=Y_i\left(0\right)\cdot e^{-a_i\cdot t}+\theta_i\cdot h\left(a_i,t\right)+\sum_{j=1}^{N}\sigma_{ij}\cdot e^{-a_i\cdot t}\cdot\int_0^t e^{a_i\cdot s}dW_j\left(s\right)$$

with

$$h(a_i,t) = \begin{cases} \frac{1-e^{-a_i \cdot t}}{a_i} & , \textit{ if } a_i > 0 \\ t & , \textit{ if } a_i = 0 \end{cases}$$

for all $t \in [0,T]$, $i \in \{1,...,N\}$. *Especially,* $Y_i(t)$ *is normally distributed for all* $t \in (0,T]$, $i \in \{1,...,N\}$ *and the expected values and covariances of* $Y_i(t)$ *and* $Y_k(t)$ *for all* $t \in (0,T]$, $i,k \in \{1,...,N\}$ *are given by*

$$E[Y_i(t)] = Y_i(0) \cdot e^{-a_i \cdot t} + \theta_i \cdot h(a_i,t)$$

and

$$Cov[Y_i(t), Y_k(t)] = h(a_i + a_k, t) \cdot \sum_{j=1}^{N} \sigma_{kj} \cdot \sigma_{lj}.$$

It should be noted that our assumption of θ, A, and σ to be constant over time may be relaxed to nonrandom, measurable, and locally bounded matrices (see, e.g., Karatzas and Shreve (1991), p. 354-355, for more details). This would allow for the modelling of deterministic dynamic drift, volatility, and mean reversion parameters by still keeping the normal distribution property. However, the resulting functional solution for Y as well as the following parameter estimation may become fairly complicated.

3 Parameter Estimation

Having defined and solved the $N-$dimensional stochastic process used to describe the evolution of the bond and stock markets we now turn to the problem of estimating the parameters of this process. To do so, we apply a two-stage procedure. First, we consider the discrete time version of equation (1). Therefore, let $\Delta t > 0$ with $T = m \cdot \Delta t$, $m \in \mathbb{N}$ denote the time distance between two observations of Y_i and $\widetilde{\zeta}(t) = \left(\widetilde{\zeta}_1(t), ..., \widetilde{\zeta}_N(t)\right)'$ with $\widetilde{\zeta}_i(t) := \widetilde{W}_i(t+\Delta t) - \widetilde{W}_i(t)$, $t \in \{0, \Delta t, ..., T-\Delta t\}$. Then, the discrete time version of equation (1) is given by

$$Y_i(t+\Delta t) - Y_i(t) = (\theta_i - a_i \cdot Y_i(t)) \cdot \Delta t + \sqrt{\Delta t} \cdot \sigma_i \cdot \widetilde{\zeta}_i(t)$$

for $i \in \{1,...,N\}, t \in \{0, \Delta t, ..., T-\Delta t\}$. If we define

$$R_i(t+\Delta t) = R_i(t, t+\Delta t) := \ln\left(\frac{S_i(t+\Delta t)}{S_i(t)}\right)$$

for each stock market index $i \in \{1,...,N\}$, with $R_i(0) := 0$, this is equivalent to

$$R_i(t+\Delta t) = \theta_i \cdot \Delta t + (1 - \widetilde{a}_i \cdot \Delta t) \cdot R_i(t) + \sqrt{\Delta t} \cdot X_i(t) \tag{3}$$

with $X_i(t) := \sigma_i \cdot \widetilde{\zeta}_i(t)$, $t \in \{0, \Delta t, ..., T-\Delta t\}$, and

$$\widetilde{a}_i := \begin{cases} a_i & , \text{if } i \text{ denotes a PEX rate} \\ \frac{1}{\Delta t} & , \text{if } i \text{ denotes a stock market index.} \end{cases}$$

Using a monthly time series of Bloomberg data for the PEX rates with maturities from one to ten years ($Y_1, ..., Y_{10}$), the DJ Euro Stoxx 50 index (Y_{11}, in Euro), and the MSCI World ex EMU index (Y_{12}, in Euro) starting from January 31, 1992 to February 28, 2001 (i.e. a sample of size $m = 109$), a least square optimization gives

$i \setminus Parameter$	θ_i (%)	a_i	σ_i (%)
1	1.4251	0.3956	0.7470
2	1.6836	0.4181	0.8624
3	1.7611	0.4064	0.8511
4	1.8363	0.4007	0.8202
5	1.8039	0.3783	0.7878
6	1.7250	0.3510	0.7502
7	1.6270	0.3238	0.7110
8	1.5239	0.2989	0.6697
9	1.4358	0.2787	0.6375
10	1.3831	0.2662	0.6160
11	18.5223	0	17.1986
12	11.4838	0	16.6912

Table 1: Parameter Estimations

us the parameter estimations[1] as stated in table 1. The resulting parameters μ_i for the stock processes are $\mu_{11} = \theta_{11} + \frac{1}{2} \cdot \sigma_{11}^2 = 20.0012\%$ and $\mu_{12} = 12.8768\%$.

For an empirical evaluation of the model we use the sample residuals

$$x_{ik} = \frac{R_i(k \cdot \Delta t) - \theta_i \cdot \Delta t - (1 - \widetilde{a}_i \cdot \Delta t) \cdot R_i((k-1) \cdot \Delta t)}{\sqrt{\Delta t}}, \; k = 1, ..., m \cdot \Delta t,$$

which have, by construction, a sample variance of

$$s_i^2 = \frac{1}{m-1} \cdot \sum_{k=1}^{m} (x_{ik} - \overline{x}_i)^2 = \sigma_i^2$$

with $\overline{x}_i = \frac{1}{m} \cdot \sum_{k=1}^{m} x_{ik}$, $i = 1, ..., N$. One method that has been suggested for testing whether the distribution underlying a sample of m elements is standard normal, commonly known as the Jarque-Bera test, uses the skewness and excess kurtosis of the standardized residuals and refers the so-called Wald statistic[2]

$$T_{i,nd} = m \cdot \left(\frac{skewness_i^2}{6} + \frac{excess\ kurtosis_i^2}{24} \right), \; i = 1, ..., N, \tag{4}$$

to the chi-squared distribution with two degrees of freedom (see, e.g., Greene (1993), p. 310, for more details or Harvey (1989), p. 260, for a similar test). The critical values for a 5%$-$ and 1%$-$significance level are 5.9915 and 9.210, respectively. The null hypothesis is rejected if $T_{i,nd}$ is greater than the corresponding critical value.

[1] Hereby, given θ_i and $\widetilde{a}_i$, the volatilities σ_i are estimated by the sample variance of the corresponding residuals $X_i(t)$, $t \in \{0, \Delta t, ..., T - \Delta t\}$, $i \in \{1, ..., N\}$.

[2] For a random variable X with finite $k-$th central moment $m_k = E\left[(X - E[X])^k\right]$, $k \in \mathbb{N}$, the skewness is given by

$$skewness = \frac{m_3}{(\sqrt{m_2})^3}$$

and the excess kurtosis by

$$excess\ kurtosis = \frac{m_4}{(\sqrt{m_2})^4} - 3.$$

If the probability density function (pdf) of X is symmetric, the skewness will be equal to 0. If X is standard normally distributed, the excess kurtosis will be equal to 0. If the excess kurtosis of X is positive, the pdf of X will have more mass in the tails than a Gaussian pdf with the same variance.

i	$skew.$	$ex.\ kurt.$	$T_{i,nd}$	$T^1_{i,ac}$	$T^2_{i,ac}$
1	-0.4452	1.2490	10.6848	3.7886	0.5445
2	-0.2256	0.6705	2.9667	3.8915	0.4779
3	0.0102	-0.2254	0.2326	3.9611	1.0040
4	0.2111	-0.4563	1.7548	3.8777	1.0654
5	0.3012	-0.4232	2.4611	3.5967	1.1808
6	0.2972	-0.5233	2.8484	3.5918	1.2645
7	0.2987	-0.5474	2.9819	3.3015	1.4457
8	0.3373	-0.5141	3.2672	3.0984	1.5350
9	0.3889	-0.4690	3.7466	2.5686	1.4202
10	0.3507	-0.4595	3.1930	2.5077	1.3748
11	-0.5585	0.7053	7.9266	0.2594	0.7984
12	-0.5579	0.1209	5.7205	0.6152	0.0167

Table 2: Statistics for testing the quality of the model

To test for autocorrelations of the residuals $\widetilde{\zeta}_i$, $i \in \{1, ..., N\}$, we apply the test statistic

$$T^l_{i,ac} = \frac{R_{i,l} \cdot \sqrt{m-2}}{\sqrt{1-R^2_{i,l}}}, \; i = 1, ..., N,$$

to the sample residuals $z_{ik} := \frac{x_{ik}}{\sigma_i}$, $k = 1, ..., m$, with $R_{i,l}$ denoting the sample autocorrelation coefficient[3] of the corresponding residuals for a lag $l \in \{1, ..., L\}$, $L < m$. Larsen and Morris (2001), p. 626, show that under the assumption (null hypothesis) that the residuals are uncorrelated, the test statistic follows a Student-t distribution with $m-2$ degrees of freedom. The critical values for a $5\%-$ and $1\%-$significance level are 1.9824 and 2.6226 (with $m = 109$), respectively. The null hypothesis is rejected if $\left|T^l_{i,ac}\right|$ is greater than the corresponding critical value.

According to table 2 we get the following results:

1. The normal distribution hypothesis is only rejected on a $5\%-$significance level for $i \in \{1, 11\}$ and on a $1\%-$significance level only for $i = 1$.

2. The zero autocorrelation hypothesis for lag 1 is rejected on a $5\%-$significance level for $i \in \{1, ..., 10\}$ and on a $1\%-$significance level for $i \in \{1, ..., 8\}$.

3. The zero autocorrelation hypothesis for lag 2 is neither rejected on a $5\%-$significance level nor on a $1\%-$significance level for all $i \in \{1, ..., 12\}$.

[3]The sample autocorrelation coefficient R_l for a lag $l \in \{0, 1, ..., L\}$, $L < m$, of a sample $(z_k)_{k=1,...,m}$ is defined by

$$R_l = \frac{m-1}{m-1-l} \cdot \frac{\sum\limits_{k=1+l}^{m} (z_k - \overline{z}) \cdot (z_{k-l} - \overline{z})}{\sum\limits_{k=1}^{m} (z_k - \overline{z})^2}$$

with

$$\overline{z} = \frac{1}{m} \cdot \sum_{k=1}^{m} z_k.$$

With respect to point 1 we may decide to neglect the one year maturity within our model. Furthermore, we may want to test an ARMA model because of the results in point 2 (see, e.g., Greene (1993), p. 550-552, for more details). However, since we would also like to use the model for pricing stock and interest rate derivatives, there is a trade-off between the best possible empirical and a good pricing model. With respect to both needs we consider the results to be sufficiently good to support the model we have chosen.

In the second step we fit the model to empirical correlation data. To do this, we use the discrete time version of equation (2) for $i \in \{1, ..., N\}, t \in [0, \Delta t, ..., T - \Delta t]$, which is given by

$$R_i(t+\Delta t) = \theta_i \cdot \Delta t + (1 - \widetilde{a}_i \cdot \Delta t) \cdot R_i(t) + \sqrt{\Delta t} \cdot \sum_{j=1}^{N} \sigma_{ij} \ \zeta_j(t) \tag{5}$$

where $\zeta(t) = (\zeta_1(t), ..., \zeta_N(t))'$ is a vector of independent standard normally distributed random variables[4]. Hence, for all $i \in \{1, ..., N\}, \ t \in [0, \Delta t, ..., T - \Delta t]$, we have

$$X_i(t) = \sum_{j=1}^{N} \sigma_{ij} \ \zeta_j(t).$$

On the other hand, for $i, k \in \{1, ..., N\}$, we get

$$\begin{aligned} Cov[X_i(t), X_k(t)] &= Cov\left[\sum_{j=1}^{N} \sigma_{ij} \ \zeta_j(t), \sum_{l=1}^{N} \sigma_{kl} \ \zeta_l(t)\right] \\ &= \sum_{j=1}^{N} \sum_{l=1}^{N} \sigma_{ij} \cdot \sigma_{kl} \cdot Cov[\ \zeta_j(t), \ \zeta_l(t)] \\ &= \sum_{j=1}^{N} \sigma_{ij} \cdot \sigma_{kj} = \sum_{j=1}^{\min\{i,k\}} \sigma_{ij} \cdot \sigma_{kj}. \end{aligned}$$

Especially,

$$\sigma_i^2 = Var[X_i(t)] = Cov[X_i(t), X_i(t)] = \sum_{j=1}^{i} \sigma_{ij}^2 > 0.$$

If we set

$$\alpha_{ij} := \frac{\sigma_{ij}}{\sigma_i}, 1 \leq j \leq i \leq N, \tag{6}$$

we get

$$Cor[X_i(t), X_k(t)] = \frac{Cov[X_i(t), X_k(t)]}{\sigma_i \cdot \sigma_k} = \sum_{j=1}^{\min\{i,k\}} \alpha_{ij} \cdot \alpha_{kj}$$

with

$$\sum_{j=1}^{i} \alpha_{ij}^2 = 1, \text{ i.e. } \alpha_{ii}^2 = 1 - \sum_{j=1}^{i-1} \alpha_{ij}^2.$$

Hence, $\alpha_{11} = 1$ and

$$\alpha_{i1} = \frac{Cor[X_i(t), X_1(t)]}{\alpha_{11}} = Cor[X_i(t), X_1(t)], \ i = 2, ..., N, \tag{7}$$

[4] We hereby assume that $\zeta_i(t)$ and $\zeta_i(t')$ are uncorrelated for all $t, t' \in [0, T]$ with $t \neq t'$.

as well as

$$\alpha_{22} = \sqrt{1-\alpha_{21}^2} = \sqrt{1-\left(Cor\left[X_2\left(t\right), X_1\left(t\right)\right]\right)^2}. \tag{8}$$

Let $i-1 \in \{2,...,N-1\}$ and suppose that we did already calculate all α_{lj}, $l=1,...,i-1$, $j=1,...,l$. Then, for $k=2,...,i-1$, we get

$$Cor\left[X_i\left(t\right), X_k\left(t\right)\right] = \sum_{j=1}^{k} \alpha_{ij}\cdot\alpha_{kj}$$

or equivalently

$$\alpha_{ik} = \frac{Cor\left[X_i\left(t\right), X_k\left(t\right)\right] - \sum_{j=1}^{k-1} \alpha_{ij}\cdot\alpha_{kj}}{\alpha_{kk}} \tag{9}$$

and

$$\alpha_{ii} = \sqrt{1-\sum_{j=1}^{i-1}\alpha_{ij}^2}. \tag{10}$$

Using equation (6) we can easily derive the corresponding values for σ_{ik}. For the above time series we get the following correlation matrix $\left(Cor\left[X_i\left(t\right), X_k\left(t\right)\right]\right)_{i,k=1,...,N}$ (in %)

$$\begin{pmatrix}
100 & 92 & 86 & 81 & 77 & 73 & 68 & 65 & 61 & 59 & -18 & -18 \\
92 & 100 & 98 & 94 & 91 & 88 & 83 & 79 & 76 & 75 & -19 & -18 \\
86 & 98 & 100 & 99 & 96 & 94 & 91 & 87 & 84 & 83 & -22 & -21 \\
81 & 94 & 99 & 100 & 99 & 98 & 95 & 93 & 90 & 89 & -24 & -23 \\
77 & 91 & 96 & 99 & 100 & 99 & 97 & 95 & 93 & 92 & -27 & -25 \\
73 & 88 & 94 & 98 & 99 & 100 & 99 & 98 & 96 & 95 & -27 & -25 \\
68 & 83 & 91 & 95 & 97 & 99 & 100 & 99 & 98 & 97 & -29 & -26 \\
65 & 79 & 87 & 93 & 95 & 98 & 99 & 100 & 99 & 99 & -28 & -25 \\
61 & 76 & 84 & 90 & 93 & 96 & 98 & 99 & 100 & 100 & -29 & -27 \\
59 & 75 & 83 & 89 & 92 & 95 & 97 & 99 & 100 & 100 & -28 & -27 \\
-18 & -19 & -22 & -24 & -27 & -27 & -29 & -28 & -29 & -28 & 100 & 76 \\
-18 & -18 & -21 & -23 & -25 & -25 & -26 & -25 & -27 & -27 & 76 & 100
\end{pmatrix}.$$

Using equations $(7)-(10)$ we iteratively derive the matrix (in %)

$$\alpha = \begin{pmatrix}
100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
92 & 38 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
86 & 47 & 19 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
81 & 51 & 25 & 15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
77 & 53 & 27 & 21 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
73 & 53 & 32 & 25 & 11 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\
68 & 53 & 37 & 27 & 13 & 15 & 11 & 0 & 0 & 0 & 0 & 0 \\
65 & 51 & 38 & 32 & 13 & 17 & 13 & 11 & 0 & 0 & 0 & 0 \\
61 & 52 & 40 & 31 & 15 & 19 & 15 & 13 & 11 & 0 & 0 & 0 \\
59 & 52 & 39 & 31 & 16 & 20 & 15 & 15 & 11 & 8 & 0 & 0 \\
-18 & -7 & -19 & -14 & -13 & 1 & -10 & 15 & -12 & -5 & 92 & 0 \\
-18 & -5 & -18 & -8 & -19 & 2 & -2 & 8 & -14 & -7 & 67 & 64
\end{pmatrix}.$$

The residuals $\zeta_i(t)$, $i \in \{1, ..., N\}$ can also be derived by an iterative procedure. Given $X_i(t)$ and $\zeta_j(t)$, $j = 1, ..., i-1$, we directly get

$$\zeta_i(t) = \frac{X_i(t) - \sum_{j=1}^{i-1} \sigma_{ij}\ \zeta_j(t)}{\sigma_{ii}}, \ i = 2, ..., N,$$

with $\zeta_1(t) = \frac{X_1(t)}{\sigma_{11}} = \frac{X_1(t)}{\sigma_1}$. Note that we can directly conclude that $\zeta(t)$ is normally distributed as soon as $X(t)$ is, $t \in [0, \Delta t, ..., T - \Delta t]$.

4 Conditional Scenarios

In this section we deal with the problem that a trader or risk manager does not want to rely on empirical information only. Research departments do make market forecasts mainly for stock market returns within a given period or interest rates for a fixed time-to-maturity and a specific point in time. Most of the time they hereby assume that the correlation and volatility structure is constant over time[5]. Consistent to such (partial) market forecasts we want to generate a complete set of market scenarios to describe the evolution of a trading book or portfolio. Therefore, let for $0 \leq t_0 \leq t \leq T$ be

$$R_i(t_0, t) = \begin{cases} R_i(t) = Y_i(t) & \text{, if } i \text{ denotes a PEX rate} \\ \ln\left(\frac{S_i(t)}{S_i(t_0)}\right) = Y_i(t) - Y_i(t_0) & \text{, else.} \end{cases}$$

Assumption 4.1 *A forecast $\overline{R}_i(t_0, t_1)$ for $R_i(t_0, t_1)$, $i \in \{1, ..., N\}$, and time interval $[t_0, t_1]$, $t_0, t_1 \in \{0, \Delta t, ..., m \cdot \Delta t\}$, defines the (conditional) stochastic process $(R_i^c(t_0, t))_{t \in [t_0, T]}$ given by*

$$R_i^c(t_0, t) = \begin{cases} R_i(t_0, t) + \frac{t - t_0}{t_1 - t_0} \cdot (\overline{R}_i(t_0, t_1) - R_i(t_0, t_1)) & , t \in [t_0, t_1] \\ R_i(t_0, t) + \overline{R}_i(t_0, t_1) - R_i(t_0, t_1) & , t \in [t_1, T]. \end{cases}$$

Now, let $t \in [t_0, t_1 - \Delta t] \cap \{0, \Delta t, ..., m \cdot \Delta t\}$ and $t_0, t_1 \in \{0, \Delta t, ..., m \cdot \Delta t\}$. Then, if i denotes a forecasted stock market index and S_i^c the corresponding conditional stock index process,

$$\begin{aligned} \Delta \ln(S_i^c(t)) &= \ln(S_i^c(t + \Delta t)) - \ln(S_i^c(t)) \\ &= \ln\left(\frac{S_i^c(t + \Delta t)}{S_i(t_0)}\right) - \ln\left(\frac{S_i^c(t)}{S_i(t_0)}\right) \\ &= R_i^c(t_0, t + \Delta t) - R_i^c(t_0, t) \\ &= R_i(t_0, t + \Delta t) - R_i(t_0, t) + \frac{\Delta t}{t_1 - t_0} \cdot (\overline{R}_i(t_0, t_1) - R_i(t_0, t_1)) \\ &= \Delta \ln(S_i(t)) + \frac{\Delta t}{t_1 - t_0} \cdot (\overline{R}_i(t_0, t_1) - R_i(t_0, t_1)). \end{aligned}$$

On the other hand, if i denotes a forecasted PEX rate, then

$$R_i^c(t) = R_i(t) + \frac{t - t_0}{t_1 - t_0} \cdot (\overline{R}_i(t_1) - R_i(t_1))$$

[5]As an extension we could allow for changing volatilities and correlations over time and include forecasts on these variables into the model. However, this may complicate the parameter estimation and significantly increase the simulation time of a portfolio. Another possibility is to simply add further full market event or crash scenarios for considering non-normal or chaotic market movements with changing volatility or correlation structure and test the portfolio behaviour under these scenarios.

and thus

$$\begin{aligned}\Delta R_i^c(t) &= R_i^c(t+\Delta t) - R_i^c(t) \\ &= R_i(t+\Delta t) - R_i(t) + \frac{\Delta t}{t_1 - t_0} \cdot (\overline{R}_i(t_1) - R_i(t_1)).\end{aligned}$$

Consequently, for each forecasted Y_i, $i \in \{1, ..., N\}$, $t \in [t_0, t_1 - \Delta t] \cap \{0, \Delta t, ..., m \cdot \Delta t\}$ and $t_0, t_1 \in \{0, \Delta t, ..., m \cdot \Delta t\}$, we have

$$\Delta Y_i^c(t) = \Delta Y_i(t) + \frac{\Delta t}{t_1 - t_0} \cdot (\overline{R}_i(t_0, t_1) - R_i(t_0, t_1)).$$

We can therefore state the following lemma.

Lemma 4.2 *A forecast* $\overline{R}_i(t_0, t_1)$ *for* $R_i(t_0, t_1)$*,* $i \in \{1, ..., N\}$*, and time interval* $[t_0, t_1]$ *with* $t_0, t_1 \in \{0, \Delta t, ..., m \cdot \Delta t\}$*, defines the discrete (conditional) stochastic process* $(Y_i^c(t))_{t \in \{0, \Delta t, ..., m \cdot \Delta t\}}$ *given by* $Y_i^c(0) = Y_i(0)$ *and*

$$\Delta Y_i^c(t) = \begin{cases} \Delta Y_i(t) & , \text{ if } t \in [0, t_0) \cup [t_1, T) \\ \Delta Y_i(t) + \frac{\Delta t}{t_1 - t_0} \cdot (\overline{R}_i(t_0, t_1) - R_i(t_0, t_1)) & , \text{ if } t \in [t_0, t_1). \end{cases}$$

Additionally, we want to ensure that for each $t \in [t_0, t_1 - \Delta t] \cap \{0, \Delta t, ..., m \cdot \Delta t\}$

$$\begin{aligned}\Delta Y_i^c(t) &= (\theta_i - a_i \cdot Y_i^c(t)) \cdot \Delta t + \sqrt{\Delta t} \cdot X_i^c(t) \\ &= \left(\theta_i - a_i \cdot \left[Y_i(t) + \frac{t - t_0}{t_1 - t_0} \cdot (\overline{R}_i(t_0, t_1) - R_i(t_0, t_1))\right]\right) \cdot \Delta t \\ &\quad + \sqrt{\Delta t} \cdot X_i^c(t)\end{aligned}$$

with $X_i^c(t)$ being the conditional random variable corresponding to $X_i(t)$ and the forecast $\overline{R}_i(t_0, t_1)$, $i \in \{1, ..., N\}$. Using lemma 4.2 we thus get

$$\begin{aligned}\Delta Y_i^c(t) &= \Delta Y_i(t) + \frac{\Delta t}{t_1 - t_0} \cdot (\overline{R}_i(t_0, t_1) - R_i(t_0, t_1)) \\ &= (\theta_i - a_i \cdot Y_i(t)) \cdot \Delta t + \sqrt{\Delta t} \cdot X_i(t) + \frac{\Delta t}{t_1 - t_0} \cdot (\overline{R}_i(t_0, t_1) - R_i(t_0, t_1)) \\ &= \left(\theta_i - a_i \cdot \left[Y_i(t) + \frac{t - t_0}{t_1 - t_0} \cdot (\overline{R}_i(t_0, t_1) - R_i(t_0, t_1))\right]\right) \cdot \Delta t \\ &\quad + \sqrt{\Delta t} \cdot X_i^c(t)\end{aligned}$$

which is equivalent to

$$X_i^c(t) = X_i(t) + \frac{1 + a_i \cdot (t - t_0)}{t_1 - t_0} \cdot \sqrt{\Delta t} \cdot \left(\overline{R}_i(t_0, t_1) - R_i(t_0, t_1)\right).$$

Now, let $\sigma\sigma'$ be the covariance matrix of the random variables $X_1, ..., X_N$, I_2 a subset of forecasted variables $k \in \{1, ..., N\}$, $I_1 = \{1, ..., N\} - I_2$, and Σ the reordered covariance matrix $\sigma\sigma'$ defined by

$$\begin{aligned}\Sigma &= \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \\ &= \begin{pmatrix} (Cov[X_i(t), X_k(t)])_{i,k \in I_1} & (Cov[X_i(t), X_k(t)])_{i \in I_1, k \in I_2} \\ (Cov[X_i(t), X_k(t)])_{i \in I_2, k \in I_1} & (Cov[X_i(t), X_k(t)])_{i,k \in I_2} \end{pmatrix}.\end{aligned}$$

Then, the conditional distribution of $(X_i^c(t))_{i \in I_1}$, given $(X_k^c(t))_{k \in I_2}$, is normal with a conditional expected value of

$$E\left[(X_i^c(t))_{i \in I_1} \mid (X_k^c(t))_{k \in I_2}\right] = \Sigma_{12}\Sigma_{22}^{-1}(X_k^c(t))_{k \in I_2}$$

and a conditional covariance of

$$\left(Cov\left[X_i^c(t), X_j^c(t) \,|\, (X_k^c(t))_{k\in I_2}\right]\right)_{i\in I_1, j\in I_1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \tag{11}$$

(see Greene (1993), p. 76-77 for a proof of this statement). Hence, for all $i \in I_1$, $t \in [t_0, t_1 - \Delta t] \cap \{0, \Delta t, ..., m \cdot \Delta t\}$ the conditional value $R_i^c(t + \Delta t)$ given $R_i^c(t)$ and $X_i^c(t)$ can be calculated as

$$R_i^c(t+\Delta t) = \theta_i \cdot \Delta t + (1 - \widetilde{a}_i \cdot \Delta t) \cdot R_i^c(t) + \sqrt{\Delta t} \cdot X_i^c(t).$$

We can therefore state the following lemma.

Lemma 4.3 *Let $\{R_k^c(t_0, t_1) : k \in I_2\}$ be a set of forecasts for $\{R_k(t_0, t_1) : k \in I_2\}$ and the time interval $[t_0, t_1]$. Furthermore, let X_k^c, $k \in I_2$, and Σ be as defined above. Then, the conditional scenario R_i^c given the forecasts and scenario R_i, $i \in I_1$, can be calculated as*

$$R_i^c(t) = \begin{cases} R_i(t) & , t \in [0, t_0] \\ \theta_i \cdot \Delta t + (1 - \widetilde{a}_i \cdot \Delta t) \cdot R_i^c(t - \Delta t) & \\ +\sqrt{\Delta t} \cdot \left(\Sigma_{12}\Sigma_{22}^{-1}\,(X_k^c(t-\Delta t))_{k\in I_2}\right)_i & , t \in [t_0 + \Delta t, t_1] \\ +\sqrt{\Delta t} \cdot X_i^{c,0}(t - \Delta t) & \\ R_i(t) + (R_i^c(t_1) - R_i(t_1)) \cdot 1_{\{i \in PEX\}}(i) & , t \in [t_1 + \Delta t, T] \end{cases}$$

where the random variables $\left(X_i^{c,0}(t)\right)_{i\in I_1}$ are normally distributed with an expected value of 0 and a covariance matrix of $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ and $\{i \in PEX\}$ is the set of all indices denoting a PEX rate.

We can now proceed as in section 3 with $\{1, ..., N\}$ substituted by I_1 and $X_i(t)$ substituted by $X_i^{c,0}(t)$, $i \in I_1$, to simulate the values for $R_i^c(t + \Delta t)$, $i \in I_1$.

Example. Using our monthly time series for the PEX rates, the DJ Euro Stoxx 50 index, and the MSCI World ex EMU index, we now look ahead in time making the forecast that there will be a 20% decrease of the DJ Euro Stoxx 50 between year one and two from now. Restarting from today with time $t = 0$, this is equivalent to $I_2 = 11$, $t_0 = 1$, $t_1 = 2$, and $R_{11}(t_0, t_1) = -20\%$. Using equation (11), the conditional standard deviations are given by

$$\begin{aligned} \left(\sigma_i^C\right)_{i\in I_1} &= \sqrt{\left(Cov\left[X_i^C(t), X_i^C(t)\right]\right)_{i\in I_1}} \\ &= \begin{pmatrix} 0.7352, 0.8469, 0.8303, 0.7953, 0.7594, 0.7222, 0.6808, \\ 0.6436, 0.6110, 0.5907, 10.8924 \end{pmatrix} \end{aligned}$$

and the conditional correlation matrix can be calculated as (in %)

$$\begin{pmatrix} 100 & 92 & 86 & 80 & 76 & 72 & 67 & 63 & 59 & 57 & -7 \\ 92 & 100 & 98 & 94 & 91 & 87 & 83 & 78 & 75 & 74 & -6 \\ 86 & 98 & 100 & 99 & 96 & 94 & 90 & 86 & 83 & 82 & -7 \\ 80 & 94 & 99 & 100 & 99 & 98 & 95 & 92 & 89 & 88 & -7 \\ 76 & 91 & 96 & 99 & 100 & 99 & 97 & 95 & 92 & 91 & -8 \\ 72 & 87 & 94 & 98 & 99 & 100 & 99 & 97 & 95 & 95 & -8 \\ 67 & 83 & 90 & 95 & 97 & 99 & 100 & 99 & 98 & 97 & -7 \\ 63 & 78 & 86 & 92 & 95 & 97 & 99 & 100 & 99 & 99 & -7 \\ 59 & 75 & 83 & 89 & 92 & 95 & 98 & 99 & 100 & 100 & -8 \\ 57 & 74 & 82 & 88 & 91 & 95 & 97 & 99 & 100 & 100 & -8 \\ -7 & -6 & -7 & -7 & -8 & -8 & -7 & -7 & -8 & -8 & 100 \end{pmatrix}.$$

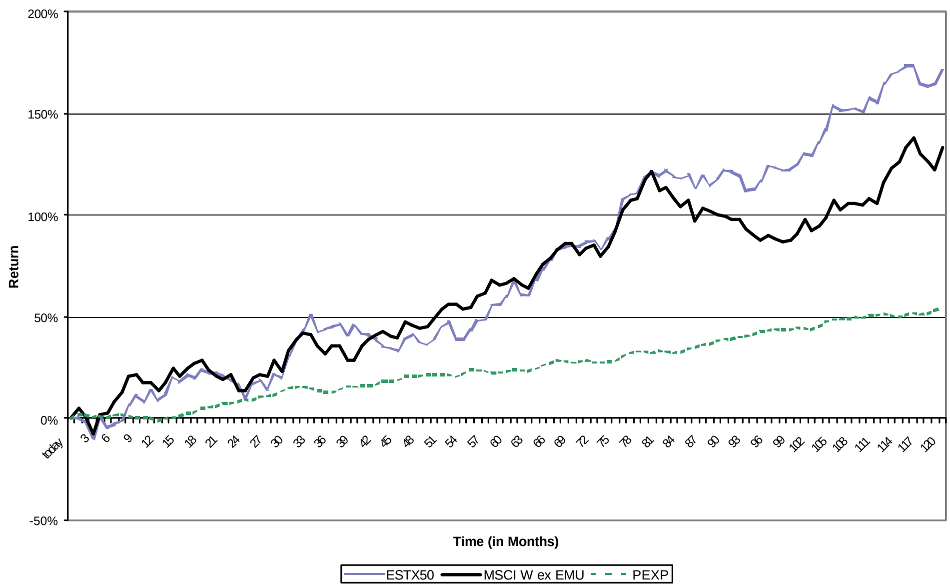

Figure 3: Typical set of unconditional scenarios for the aggregated log-return of the PEXP, DJ Euro Stoxx 50, and the MSCI World ex EMU index

Using equations $(7)-(10)$ we iteratively derive the matrix α^C as described in section 3. Figures 3 and 4 show a typical set of unconditional and conditional scenarios, based on a monthly time grid, for the different indices as they will be used to decide for a specific portfolio composition in the next section.

5 Risk Management

One of the most important ingredients to a successful risk management process is the creation of a sufficiently good set of risk numbers. These numbers should give the trader or risk manager a rather complete information on the risk of his trading book or portfolio. Beside the usually reported numbers expected return and standard deviation, we apply the so-called lower partial moments to consider the downside risk of a portfolio. Therefore, we use the generated scenario sets to calculate risk numbers according to unconditional and forecasted market movements. Given the scenarios $\mathsf{F}^k = \left(R_1^k, ..., R_N^k\right)$ and $\mathsf{F}^{c,k} = \left(R_1^{c,k}, ..., R_N^{c,k}\right)$, $k = 1, ..., K \in I\!N$, we calculate the simulations for the future values (returns)

$$V_i^k(t) = V_i\left(\mathsf{F}^k, t\right),\ k = 1, ..., K,$$

for the PEXP ($i = 1$), the DJ Euro Stoxx 50 ($i = 2$), and the MSCI World ex EMU ($i = 3$). For any portfolio $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ which we consider to be fixed from 0 to the end of the planning horizon T for the ease of exposition, the future value $V(\varphi, t)$ of this portfolio at time t is given by the random variable

$$V(\varphi, t) = \sum_{i=1}^{N} \varphi_i \cdot V_i(t)$$

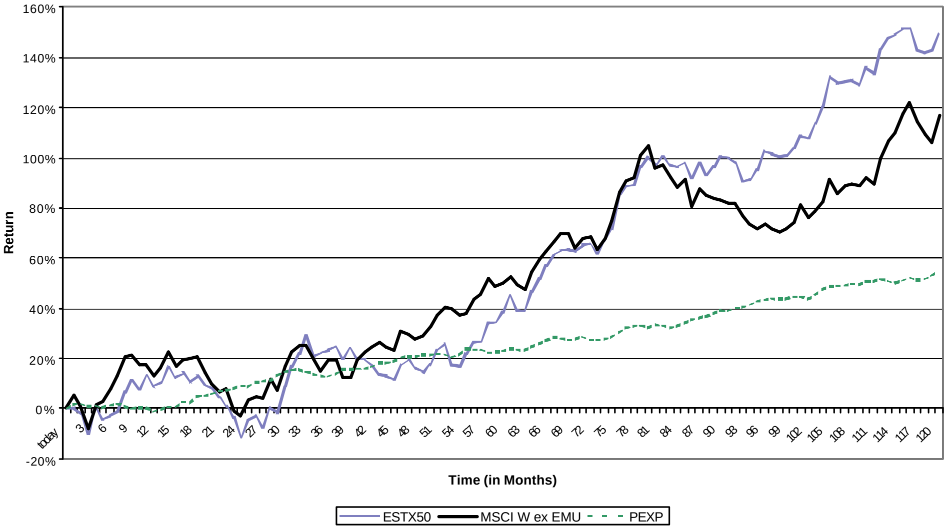

Figure 4: Typical set of conditional scenarios for the aggregated log-return of the PEXP, DJ Euro Stoxx 50, and the MSCI World ex EMU index

with $V(\varphi, 0)$ denoting the portfolio value at time 0. Using our simulations, the future value $V(\varphi, t)$ is simulated by

$$V^k(\varphi, t) = \sum_{i=1}^{N} \varphi_i \cdot V_i^k(t), \; k = 1, ..., K.$$

To measure the downside risk of the future value at time t we consider the discrete version of the lower partial moment of order $l \in I\!N$ corresponding to an investor specific benchmark (return) $B(t) \in I\!R$ which is defined by

$$LPM_l(\varphi, V, B, t) = \sum_{\substack{k=1,...,K \\ V^k(\varphi,t)<B(t)}} p_k \cdot \left(B(t) - V^k(\varphi, t)\right)^l. \qquad (12)$$

The lower partial moment only considers realizations of the future value of V below the investor specific benchmark measured to a power of l. For $l = 0$ this is the probability that the random future value falls below the given benchmark which is referred to as shortfall probability. Setting the benchmark equal to 0 reflects the probability of loss. For $l = 1$, the lower partial moment is the expected deviation of the future values below the benchmark, sometimes called (expected) regret. For $l = 2$, the lower partial moment is weighting the squared deviations below the benchmark and thus is the semi-variance if the benchmark is set equal to the expected future value. For a more detailed discussion of the lower partial moments see, e.g., Harlow (1991) or Zagst (2002). It should be noted that the lower partial moment of order 0 is closely related to the value at risk (VaR) of a portfolio or trading book. The VaR of a portfolio φ, given a shortfall probability $\alpha \in (0, 1)$ and a time horizon t, considers the potential change in the portfolio value between times 0 and t. It is defined to be the difference of the expected change in the portfolio value and the benchmark $B(t)$ at which the lower partial moment of order 0 first crosses the level

α. For an overview on the different methods to calculate the VaR of a portfolio see, e.g., Wilson (1996), Smithson and Minton (1996), or Zagst (1997). In this section we will use the following criteria to decide for a specific portfolio composition:

1. The expected return over a five year planning horizon should be maximized under normal market conditions (unconditional scenarios).
2. At each $t \in \{1y, ..., 5y\}$ the expected return under the given forecast (conditional scenarios) should be at least 2.5% p.a.
3. At each $t \in \{1y, ..., 5y\}$ the probability of getting a portfolio return of at least 4% under normal market conditions should be greater than 75%. Furthermore, the probability of loss under the given forecast is limited by 30%.
4. We want to choose between the three portfolios $\varphi^1 = (40\%, 30\%, 30\%)$, $\varphi^2 = (50\%, 25\%, 25\%)$ and $\varphi^3 = (60\%, 20\%, 20\%)$ today under the assumption that we plan to hold the portfolio until the end of the planning horizon (in 5 years).

Using the (conditional) scenarios created as described in the previous section, we get the following information for the given portfolios (numbers under conditional scenarios in brackets):

φ^1	$1y$	$2y$	$3y$	$4y$	$5y$
Expected return p.a.	12.24 (12.24)	11.99 (1.65)	12.07 (4.88)	12.21 (6.60)	12.45 (7.77)
$P(R(\varphi^1, 0, t) < 4\%)$	21.10 (21.10)	15.00 (69.30)	10.30 (43.60)	6.30 (30.30)	4.60 (19.10)
$P(R(\varphi^1, 0, t) < 0\%)$	11.90 (11.90)	5.50 (38.00)	2.30 (15.70)	0.90 (5.20)	0.40 (2.20)

φ^2	$1y$	$2y$	$3y$	$4y$	$5y$
Expected return p.a.	11.17 (11.17)	10.98 (2.27)	11.07 (4.96)	11.22 (6.40)	11.46 (7.40)
$P(R(\varphi^2, 0, t) < 4\%)$	20.90 (20.90)	14.90 (67.50)	10.10 (42.10)	5.90 (28.60)	4.20 (17.50)
$P(R(\varphi^2, 0, t) < 0\%)$	9.80 (9.80)	3.80 (28.90)	1.10 (11.20)	0.70 (3.00)	0.10 (1.10)

φ^3	$1y$	$2y$	$3y$	$4y$	$5y$
Expected return p.a.	10.07 (10.07)	9.93 (2.87)	10.01 (5.02)	10.17 (6.18)	10.40 (7.00)
$P(R(\varphi^3, 0, t) < 4\%)$	20.80 (20.80)	14.20 (64.00)	9.70 (39.70)	5.90 (27.00)	3.50 (15.40)
$P(R(\varphi^3, 0, t) < 0\%)$	8.20 (8.20)	2.50 (20.10)	0.70 (5.00)	0.30 (0.90)	0.10 (0.20)

The only portfolio holding all conditions required is portfolio φ^3. We therefore decide for an investment of 60% in the interest rate market and of 20% in each of the two stock markets. It should be noted that the choice of the optimal portfolio was rather easy because we did only allow for a finite set of possible portfolios. If the set of possible portfolios is continuous, we can apply a mixed-integer optimization program to solve the previously described problem (see, e.g., Zagst (2002) for more details). For the investor it is important to understand the implications of a chosen or proposed portfolio strategy on a strategic time horizon. Therefore, he is interested in evaluating potential portfolio developments and their characteristics at different future points in time. For the ease of exposition we did show the implications of a simple buy-and-hold strategy. More complex strategies, e.g. rebalancing strategies or the inclusion of stop-and-loss decision rules, can easily be incorporated in the presented concept but need more detailed information on the investor's preferences and constraints.

6 Conclusion

We introduced a multi-factor market model to describe the evolution of interest rates and stock indices which may also be used for pricing financial derivatives. The model was solved and fitted to empirical market data and used for generating scenarios on stock and bond market indices. We showed how the specific forecasts made by researchers and traders can be integrated to derive conditional scenarios for the whole universe considered. These scenarios were then applied to the management of a portfolio or trading book under given limits for the downside risk and the expected performance of the resulting portfolio.

7 Appendix

Proof of theorem 2.1. Consider the deterministic matrix differential equation

$$\dot{\Phi}(t) = A\Phi(t), \ \ \Phi(0) = I_N$$

where I_N denotes the $N \times N$ identity matrix. The solution of this deterministic matrix differential equation is given by

$$\Phi(t) = \begin{pmatrix} e^{-a_1 \cdot t} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & e^{-a_N \cdot t} \end{pmatrix}.$$

Using Karatzas and Shreve (1991), p. 354-355, the solution of the stochastic differential equation (2) is given by

$$Y(t) = \Phi(t)\left[Y(0) + \int_0^t \Phi^{-1}(s)\,\theta ds + \int_0^t \Phi^{-1}(s)\,\sigma dW(s)\right]$$

with

$$m(t) = E[Y(t)] = \Phi(t)\left[Y(0) + \int_0^t \Phi^{-1}(s)\,\theta ds\right]$$

and

$$\begin{aligned} C(t) &= (Cov[Y_i(t), Y_k(t)])_{\substack{i=1,\ldots,N \\ k=1,\ldots,N}} \\ &= \Phi(t)\left[\int_0^t \Phi^{-1}(s)\,\sigma\left(\Phi^{-1}(s)\,\sigma\right)' ds\right](\Phi(t))'. \end{aligned}$$

Hence,

$$\begin{aligned} Y_i(t) &= e^{-a_i \cdot t} \cdot \left[Y_i(0) + \int_0^t \theta_i \cdot e^{a_i \cdot s} ds + \sum_{j=1}^N \int_0^t e^{a_i \cdot s} \sigma_{ij} dW_j(s)\right] \\ &= Y_i(0) \cdot e^{-a_i \cdot t} + \theta_i \cdot h(a_i, t) + \sum_{j=1}^N \sigma_{ij} \cdot e^{-a_i \cdot t} \cdot \int_0^t e^{a_i \cdot s} dW_j(s) \end{aligned}$$

with

$$h(a_i, t) = \begin{cases} \frac{1 - e^{-a_i \cdot t}}{a_i} & \text{, if } a_i > 0 \\ t & \text{, if } a_i = 0. \end{cases}$$

Furthermore,

$$m_i(t) = Y_i(0) \cdot e^{-a_i \cdot t} + \theta_i \cdot h(a_i, t).$$

Using

$$\Phi^{-1}(t)\,\sigma = \begin{pmatrix} \sigma_{11} \cdot e^{a_1 \cdot t} & \cdot & \cdot & \cdot & \sigma_{1N} \cdot e^{a_1 \cdot t} \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot \\ \sigma_{N1} \cdot e^{a_N \cdot t} & \cdot & \cdot & \cdot & \sigma_{NN} \cdot e^{a_N \cdot t} \end{pmatrix}$$

we get

$$\begin{aligned} \int_0^t \Phi^{-1}(s)\,\sigma\left(\Phi^{-1}(s)\,\sigma\right)' ds &= \left(\int_0^t \sum_{j=1}^N \sigma_{ij} \cdot \sigma_{kj} \cdot e^{(a_i + a_k)\cdot s} ds\right)_{\substack{i=1,\dots,N \\ k=1,\dots,N}} \\ &= \left(h(a_i + a_k, t) \cdot \sum_{j=1}^N \sigma_{ij} \cdot \sigma_{kj}\right)_{\substack{i=1,\dots,N \\ k=1,\dots,N}} \end{aligned}$$

and thus

$$\begin{aligned} C(t) &= \Phi(t)\left(h(a_i + a_k, t) \cdot \sum_{j=1}^N \sigma_{ij} \cdot \sigma_{kj}\right)_{\substack{i=1,\dots,N \\ k=1,\dots,N}} (\Phi(t))' \\ &= \Phi(t)\left(\sum_{k=1}^N h(a_i + a_k, t) \cdot \sum_{j=1}^N \sigma_{ij} \cdot \sigma_{kj} \cdot \Phi_{kl}(t)\right)_{\substack{i=1,\dots,N \\ l=1,\dots,N}} \\ &= \Phi(t)\left(h(a_i + a_k, t) \cdot \sum_{j=1}^N \sigma_{ij} \cdot \sigma_{lj} \cdot e^{-a_l \cdot t}\right)_{\substack{i=1,\dots,N \\ l=1,\dots,N}} \\ &= \left(\sum_{i=1}^N \Phi_{ki}(t) \cdot h(a_i + a_k, t) \cdot \sum_{j=1}^N \sigma_{ij} \cdot \sigma_{lj} \cdot e^{-a_l \cdot t}\right)_{\substack{k=1,\dots,N \\ l=1,\dots,N}} \\ &= \left(h(a_i + a_k, t) \cdot \sum_{j=1}^N \sigma_{kj} \cdot \sigma_{lj} \cdot e^{-(a_k + a_l)\cdot t}\right)_{\substack{k=1,\dots,N \\ l=1,\dots,N}} \\ &= \left(h(a_i + a_k, t) \cdot \sum_{j=1}^N \sigma_{kj} \cdot \sigma_{lj}\right)_{\substack{k=1,\dots,N \\ l=1,\dots,N}}. \end{aligned}$$

□

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