Pricing of Spread Options on stochastically correlated underlyings

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Abstract

This paper proposes a method to price spread options on stochastically correlated underlying assets. Therefore it provides a more realistic approach towards correlation structure. We generalize a constant correlation tree model developed by Hull (2002) and extend it by the notion of stochastic correlation. The resulting tree model is recombining and easy to implement. Moreover, the numerical convergence of our model is very fast. Our sensitivity analysis with respect to the stochastic correlation parameters shows that the constant correlation model systematically overprices spread options on two stochastically correlated underlying assets. Furthermore, we use our model to derive hedging parameters for the correlation of a spread option and show that the constant correlation model also overprices the hedging parameters.

Key words: stochastic correlation, spread options, pricing, Greeks, binomial tree, trinomial tree

JEL classification: G13, C63

1 Introduction

A spread option is a derivative on the difference of two underlying assets with a terminal payoff of the form $[S_1(T) - S_2(T) - K]^+$, where $S_1(T)$ and $S_2(T)$ denote the values of the underlying assets in T and K the exercise price. The main challenge in pricing spread options lies in the lack of knowledge about the distribution of the difference between two non-trivially correlated stochastic processes (see Dempster and Hong (2000)). Among the different approaches to spread option pricing is the arithmetic Brownian motion model, in which the prices of the underlyings as well as the spread are modeled by Brownian motions with constant correlations (see Poitras (1998)). This setting allows a closed form solution but does not prevent negative values for the underlyings. Other approaches like Carmona and Durrleman (2003), Pearson (1995) or Shimko (1994) model the underlying assets as geometric Brownian motions assuming constant correlation. In a recent paper, Dempster and Hong (2000), introduces a reasonably fast numerical method to price these derivatives under a framework of stochastic volatility. They play with the idea of stochastic correlation but in general, no publications, to the best of our knowledge, has introduced a stochastic covariance structure for the underlying assets in their pricing models.

In plain vanilla option pricing the assumptions of the Black-Scholes model on volatility have been relaxed by works of Hull and White (1987, 1988), Stein and Stein (1991), Heston (1993) and Shu and Zhang (2000). However, so far the correlation structure has been hardly addressed even though there are many papers which find evidence for stochastically changing correlations. Among more recent papers Ramchand and Susmel (1998) use a switching ARCH technique to find evidence for differences in correlations across variance regimes. Ball and Torous (2000) show for their data from international stock markets that the estimated correlation structure is dynamically changing over time. Before, Makridakis and Wheelwright (1974) found that international correlations are unstable over time and Kaplanis (1988) rejected the null hypothesis of constant correlations comparing matrices of monthly returns of ten markets. But also within a single market correlations seem to change stochastically, which can be seen from the correlations computed for a 50 days time window on the time series of IBM, GM and Cisco stocks from 1986 to 2006 (see Figure 1). The stochastic nature is evident but, as expained before, so far in literature there are only impulses

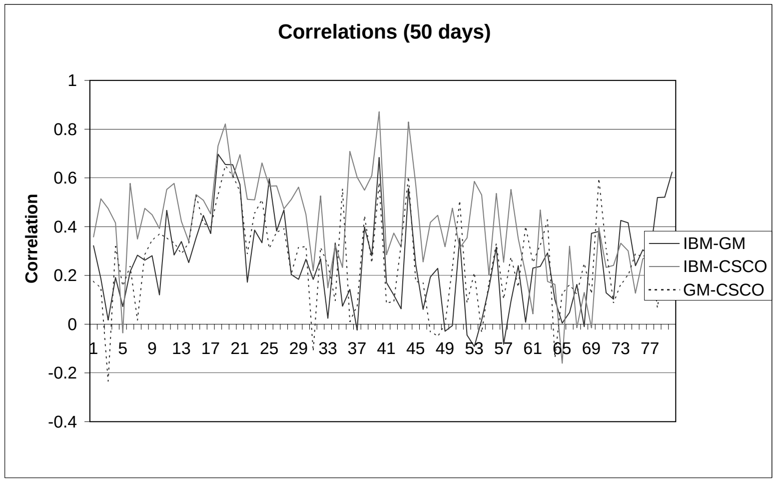

Figure 1: Structure of the correlations for a window of 50 days

and suggestions by Dempster and Hong (2000) and Dupire (1993) how to handle stochastic correlation.
In this paper we want to relax the assumption of constant correlation most of the existing literature concerning spread option pricing makes. We price spread options on stochastically correlated underlying assets using a bivariate binomial tree model. The tree model generalizes a constant correlation tree model developed by Hull (2002) and extends it by the notion of stochastic correlation. Hull's constant correlation tree model does not impose any restrictions on the correlation structure which eases the introduction of stochastic correlation. The advantage of the Hull method is that the tree is recombining because the increments of the up and down jumps of the singular assets are independent from the correlation structure. Thus, despite of the introduction of stochastic correlation, our method is easy to implement and the numerical convergence is very speedy. This stochastic correlation model allows for a more realistic approach towards correlation structure. Our sensitivity analysis with respect to the stochastic correlation parameters shows that the Hull

constant correlation model systematically overprices spread options on two stochastically correlated underlying assets. Furthermore, we provide more realistic hedging parameters for the correlation of a spread option priced with our method.

We propose a structure for the underlying processes in Section 2. In Section 3 the bivariate binomial tree model for constant correlation is derived in detail. Section 4 describes the numerical approximation of the stochastic correlation using trinomial trees. We combine the numerical approximation of the underlyings and the stochastic correlation in Section 5. Section 6 analyzes the sensitivity of the price of the spread option with respect to the parameters of the stochastic correlation process and provides the hedging parameters for the spread option. We will conclude in Section 7.

2 Underlying Processes

To model the correlation we propose a transformation $y(t)$ of the correlation, which maps its distribution from $[-1;1]$ to $(-\infty,\infty)$.[1] We found that the real correlation data under this transformation followed a mean reverting process. The system of processes is defined on a filtered probability space $(\Omega, \mathcal{F}, \tilde{\mathcal{Q}}, \mathbb{F})$ where $\mathcal{F}_0$ contains all subsets of the $(\tilde{\mathcal{Q}}-)$ null sets of $\mathcal{F}$ and $\mathbb{F}$ is right-continuous. As we assume the market to be complete the processes are defined under the risk neutral measure $\tilde{\mathcal{Q}}$. We propose the following system of underlying processes:

$$
\begin{aligned}
dS_i &= S_i r dt + S_i \sigma_i dW_i \quad \text{for} \quad i \in \{1,2\} && (1)\\
\rho(y) &= 1 - 2\exp(-\exp(y_t)) && (2)\\
dy &= a(b - y_t)dt + c dZ, && (3)\\
\text{where} & && (4)\\
& E[dW_1 dW_2 | \mathcal{F}_t] = \rho(t)dt, && (5)\\
& E[dW_i dZ] = 0. && (6)
\end{aligned}
$$

S_i are the prices of the two stocks, σ_i, r, a, b and, c are fixed constants, and dW and dZ are Wiener processes. dW and dZ are independent. The correlation

[1] We applied several transformations to the data. The here proposed transformation fitted the data best in terms of deviation from the model assumptions of gaussianity for $y(t)$.

is governed by an arithmetic Ornstein-Uhlenbeck process, with a tendency to revert back to a long-run average level of b.

3 Binomial Tree Model for two Assets with Constant Correlation

For the construction of the model with constant correlation, the assets are assumed to follow a geometric Brownian motion with constant drift and volatility, $dS_i = S_i\mu_i dt + S_i\sigma_i dW_i,\ i \in \{1,\ 2\}$ (see (1)).
Therefore $S_i(t) = S_i(0)exp(r - \frac{1}{2}\sigma_i^2)t + \sigma_i W_i(t)$. The constant correlation is defined by $E[dW_1 dW_2|\mathcal{F}_t] = \rho dt$. For the binomial approximation the lifetime of the option is divided in $n = \frac{T}{\Delta t}$ equal time steps, where Δt is the length of one time step. It is assumed that both assets can jump to two different values at each time step: The assets can increase after one time step by u_i (u_j) with probability p_i (p_j) or fall by d_i (d_j) with $1-p_i$ ($1-p_j$) respectively. Thus, if $S_1(t)$ and $S_2(t)$ are the values of the two assets at time step t then the values of $S_1(t+1)$ and $S_2(t+1)$ can be any of the combinations

$$
\begin{array}{llll}
u_1 S_1 & u_2 S_2 & \text{with probability} & p_a \\
u_1 S_1 & d_2 S_2 & \text{with probability} & p_b \\
d_1 S_1 & u_2 S_2 & \text{with probability} & p_c \\
d_1 S_1 & d_2 S_2 & \text{with probability} & p_d, \\
\text{with} & & &
\end{array}
$$

$$p_a + p_b + p_c + p_d = 1 \tag{7}$$
$$p_a + p_b = p_1 \tag{8}$$
$$p_d + p_c = 1 - p_1 \tag{9}$$
$$p_b + p_d = 1 - p_2 \tag{10}$$
$$p_a + p_c = p_2 \tag{11}$$

The nodes in the tree are denoted by (i, j, t), where i and j indicate the number of upwards moves of the first and second asset respectively and t the time ($t\Delta t$) that has passed since $t = 0$. Thus, in a recombining tree the possible number of combinations of the stock prices after a jump at time t is $(t+1)^2$. This interrelationship between the number of time steps and combinations ensures that the numerical algorithm is not exponentially dependent in time.

Proposition 1. *(Bidimensional binomial approximation)*
The conditions for the Bidimensional binomial model are given by

$$e^{r\Delta t} = u_1 p_1 + (1-p_1)d_1 \tag{12}$$
$$e^{r\Delta t} = u_2 p_2 + (1-p_2)d_2 \tag{13}$$
$$e^{r\Delta t}(u_1+d_1) - u_1 d_1 - e^{2r\Delta t} = \sigma_1^2 \Delta t \tag{14}$$
$$e^{r\Delta t}(u_2+d_2) - u_2 d_2 - e^{2r\Delta t} = \sigma_2^2 \Delta t \tag{15}$$
$$\begin{aligned} & u_1 u_2 p_a + u_1 d_2 p_b + d_1 u_2 p_c \\ & + d_1 d_2 p_d - (u_1 p_1 + (1-p_1)d_1)(u_2 p_2 + (1-p_2)d_2) = \sigma_1 \sigma_2 \rho \Delta t. \end{aligned} \tag{16}$$

For a proof see Appendix (A).

Proposition 2.
The correlation ρ is restricted by the following conditions:

$$\frac{(p_1p_2-1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2 p_1 p_2 \Delta t} \leq \rho \leq \frac{(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2\Delta t}$$
$$\frac{p_1(p_2-1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2 p_1 p_2 \Delta t} \leq \rho \leq \frac{(1-p_1(1-p_2))(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{p_1 p_2 \sigma_1\sigma_2\Delta t}$$
$$\frac{p_2(1-p_1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2 p_1 p_2 \Delta t} \leq \rho \leq \frac{(1-p_2(1-p_1))(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{p_1 p_2 \sigma_1\sigma_2\Delta t}$$
$$\frac{((1-p_1)(1-p_2)-1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2 p_1 p_2 \Delta t} \leq \rho \leq \frac{(1-p_1)(1-p_2)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{p_1 p_2 \sigma_1\sigma_2\Delta t}$$

For a proof see Appendix (B).

If the Cox Ross Rubinstein model (1979) is used the correlation has to be restricted to a proper subset of the interval $[-1,1]$.[2] In the following, we choose $p_i = 0.5$ as stated in the following proposition.

Proposition 3. *If $p_i = 0.5$, the correlations are not constrained.*

[2] The Cox Ross Rubinstein one-dimensional tree model (1979) specifies $u_i = e^{\sigma_i\sqrt{\Delta t}}$,

Proof.
From Equations (12) and (13) we get

$$2e^{r\Delta t} = u_1 + d_1, \quad 2e^{r\Delta t} = u_1 + d_1. \tag{17}$$

Equation (14) reduces to

$$u_i^2 - 2e^{r\Delta t}u_i + e^{2r\Delta t} - \sigma_i^2\Delta t = 0.$$

This is solved by

$$u_i = e^{r\Delta t} + \sigma_i\sqrt{\Delta t}, \quad d_i = e^{r\Delta t} - \sigma_i\sqrt{\Delta t}, \quad i \in \{1, 2\}. \tag{18}$$

Substituting Equations (18) in (42) it can be shown that

$$p_a = \frac{1}{4} + \frac{1}{4}\rho, \quad p_b = \frac{1}{4} - \frac{1}{4}\rho, \quad p_c = \frac{1}{4} - \frac{1}{4}\rho, \quad p_d = \frac{1}{4} + \frac{1}{4}\rho \tag{19}$$

and it follows that the probabilities are positive for $-1 \leq \rho \leq 1$. □

4 Numerical Implementation of the Mean-reverting Process

The process (3) is implemented using the trinomial tree suggested by Hull and White (1990). In the following, nodes are denoted by (l, t), where l is the number of upwards movements, i.e. the value $y(l, t) = y(0) + l\Delta y$, and t indicates the number of time steps passed since $t = 0$. For the implementation of (3) the three branching methods illustrated in Figures 2-4 are applied, where $\kappa = l$, $\kappa = l + 1$ and $\kappa = l - 1$ respectively.

$d_i = e^{-\sigma_i\sqrt{\Delta t}}$, $p_i = \frac{1}{2} + \frac{1}{2}\frac{r}{\sigma_i}\sqrt{\Delta t}$. In this case ρ is restricted by

$p_a : -4p_1p_2 \leq \rho \leq 4(1 - p_1p_2)$
$p_b : 4(p_1(1 - p_2) - 1) \leq \rho \leq 4p_1(1 - p_2)$
$p_c : 4(p_2(1 - p_1) - 1) \leq \rho \leq 4p_2(1 - p_1)$
$p_d : -4(1 - p_1)(1 - p_2) \leq \rho \leq 4(1 - (1 - p_1)(1 - p_2))$
where

$$p_i = \frac{1}{2} + \frac{1}{2}\frac{r}{\sigma_i}\sqrt{\Delta t}.$$

$$\begin{tikzcd}[row sep=small] & (l+1,t+1) \\ (l,t) \arrow[ur] \arrow[r] \arrow[dr] & (l,t+1) \\ & (l-1,t+1) \end{tikzcd}$$

Figure 2: Relation between time step t and $t+1$ when $\kappa = l$

$$\begin{tikzcd}[row sep=small] & (l+2,t+1) \\ & (l+1,t+1) \\ (l,t) \arrow[uur] \arrow[ur] \arrow[r] & (l,t+1) \end{tikzcd}$$

Figure 3: Relation between time step t and $t+1$ when $\kappa = l+1$

$$\begin{tikzcd}[row sep=small] (l,t) \arrow[r] \arrow[dr] \arrow[ddr] & (l,t+1) \\ & (l-1,t+1) \\ & (l-2,t+1) \end{tikzcd}$$

Figure 4: Relation between time step t and $t+1$ when $\kappa = l-1$

The probabilities are derived by matching the first two moments to the continuous distribution (see Appendix C).[3]

$$
\begin{aligned}
p_{l,\kappa+1} &= \frac{c^2\Delta t}{2(\Delta y)^2} + \frac{\eta^2}{2(\Delta y)^2} + \frac{\eta}{2\Delta y} && (20)\\
p_{l,\kappa} &= 1 - \frac{c^2\Delta t}{(\Delta y)^2} - \frac{\eta^2}{(\Delta y)^2} && (21)\\
p_{l,\kappa-1} &= \frac{c^2\Delta t}{2(\Delta y)^2} + \frac{\eta^2}{2(\Delta y)^2} - \frac{\eta}{2\Delta y}, && (22)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta &= \mu(l,t)\Delta t + (l-\kappa)\Delta y,\\
\mu(l,t) &= a(b - y(l,t))
\end{aligned}
$$

For a proof see Hull and White (1990). When Δy is set to $c\sqrt{3\Delta t}$ the following dynamic rules for the choice of κ can be implemented to ensure positive probabilities (see Appendix D):

$$
\kappa = \begin{cases} l+1 & \text{if} \quad \frac{\mu(l,t)\Delta t}{\Delta y} \geq \sqrt{\frac{2}{3}} \\ l & \text{if} \quad -\sqrt{\frac{2}{3}} < \frac{\mu(l,t)\Delta t}{\Delta y} < \sqrt{\frac{2}{3}} \\ l-1 & \text{if} \quad \frac{\mu(l,t)\Delta t}{\Delta y} \leq -\sqrt{\frac{2}{3}} \end{cases} \tag{23}
$$

These dynamic rules of choice for κ imply minimum and maximum values for $y(l,t)$:

$$
-\sqrt{\frac{2}{3}} \leq a(b-y(l,t))\frac{\Delta t}{\Delta y} \leq \sqrt{\frac{2}{3}} \Leftrightarrow y_{min} = b - \sqrt{\frac{2}{3}}\frac{\Delta y}{a\Delta t} \leq y(l,t) \leq b + \sqrt{\frac{2}{3}}\frac{\Delta y}{a\Delta t} = y_{max}
$$

[3] The probabilities could also be derived by converting the underlying differential equation into a set of difference equations by the explicit finite difference method. In this case the η^2 terms can be skipped. However, the procedure with the quadratic terms ensured better numerical convergence when we tested it.

The branching method is changed to $\kappa = l - 1$ at a node (n, t), where n is the largest integer with $y = y(0) + n\Delta y \leq y_{max}$ and to $\kappa = l + 1$ at a node (m, t), where m is the smallest integer with $y = y(0) + m\Delta y \geq y_{min}$.
As $y(l, t)$ has a range of $(-\infty, \infty)$ we impose the following restrictions on the product ab:

$$y_{min} = b - \sqrt{\frac{2}{3}}\frac{\Delta y}{a\Delta t} << 0 \Leftrightarrow \sqrt{2}\frac{c}{\sqrt{\Delta t}} >> ab \tag{24}$$

$$y_{max} = b + \sqrt{\frac{2}{3}}\frac{\Delta y}{a\Delta t} >> 0 \Leftrightarrow -\sqrt{2}\frac{c}{\sqrt{\Delta t}} << ab \tag{25}$$

5 Binomial Tree Model for two Assets with Stochastic Correlation

To approximate the system proposed in Section 2 we combine the two tree models introduced in Sections 3 and 4. The nodes in the combined tree are denoted by (i, j, l, t), where i and j indicate the number of up or down moves of the first and the second asset respectively as well as l specifies the level of the correlation that influences the probability structure of the movements of the assets in $t + 1$. As the correlations are not constrained in the binomial tree model in Section 3 the transformation (2) and the process for the transformation (3) of the stochastic correlation do not have to be restricted and the tree approximations for the processes of two constantly correlated assets and for the stochastic correlation can be combined without any restriction. The two trees are arranged successively in such way that the correlations $\rho_{l,t}$ resulting from the approximation of the stochastic correlation in time step t have an impact on the probabilities for an up or down jump of the assets in $t + 1$. The probabilities derived for the movements of the assets (19) also apply in the case of stochastic correlation. Furthermore, as we assume the Brownian motions of the underlying processes of the assets and of the transform of the correlation to be independent, their probabilities can be simply multiplied to obtain the joint probability. Thus, a particular node branches in 12 different nodes in the next time step.
The nodes and their probabilities are specified in Table 1. The first column encloses all 12 possible branches from a single node (i, j, l, t), while the second column provides the probability of getting to the particular node as the

product of $p_{x,y}$ and $p_{l,y,t}$ (where $x \in [a,b,c,d]$ and $y \in [l-1,l,l+1]$). The structure of the tree is illustrated in Figure 5, where the matrices in the second part of the Figure describe the possible values of S_1 and S_2.

Table 1: Nodes and Probabilities of the combined tree

Nodes	Probability
$(i+1,j+1,l+1,t+1)$	$p_{a,l+1} \cdot p_{l,l+1,t}$
$(i+1,j-1,l+1,t+1)$	$p_{b,l+1} \cdot p_{l,l+1,t}$
$(i-1,j+1,l+1,t+1)$	$p_{c,l+1} \cdot p_{l,l+1,t}$
$(i-1,j-1,l+1,t+1)$	$p_{d,l+1} \cdot p_{l,l+1,t}$
$(i+1,j+1,l,t+1)$	$p_{a,l} \cdot p_{l,l,t}$
$(i+1,j-1,l,t+1)$	$p_{b,l} \cdot p_{l,l,t}$
$(i-1,j+1,l,t+1)$	$p_{c,l} \cdot p_{l,l,t}$
$(i-1,j-1,l,t+1)$	$p_{d,l} \cdot p_{l,l,t}$
$(i+1,j+1,l-1,t+1)$	$p_{a,l-1} \cdot p_{l,l-1,t}$
$(i+1,j-1,l-1,t+1)$	$p_{b,l-1} \cdot p_{l,l-1,t}$
$(i-1,j+1,l-1,t+1)$	$p_{c,l-1} \cdot p_{l,l-1,t}$
$(i-1,j-1,l-1,t+1)$	$p_{d,l-1} \cdot p_{l,l-1,t}$

6 Sensitivity Analysis and Comparison to the Hull two-dimensional constant correlation Model

Pricing a spread option in this framework involves a considerable number of input parameters. In the following, we want to stress on the influence of the parameters of the stochastic correlation on the price of a spread option with a payoff $max(S_1 - S_2 - K, 0)$. We define the following basic scenario:

Basic Scenario: $r = 0.04$, Maturity = 1 year, $\Delta t = \frac{1}{n}$, n = Number of Time Steps, $S_0^1 = 1$, $S_0^2 = 1$, $K = 0$, $\sigma_1 = 0.3$, $\sigma_2 = 0.13$, $\rho(t=0) = 0$, a=1, b = $\ln(ln(2))$ (equivalent to $\rho = 0$), $c = 0.2$

Furthermore, we compare our Stochastic Correlation Model (SC-Model) to the Hull two-dimensional constant correlation model and show that the Hull

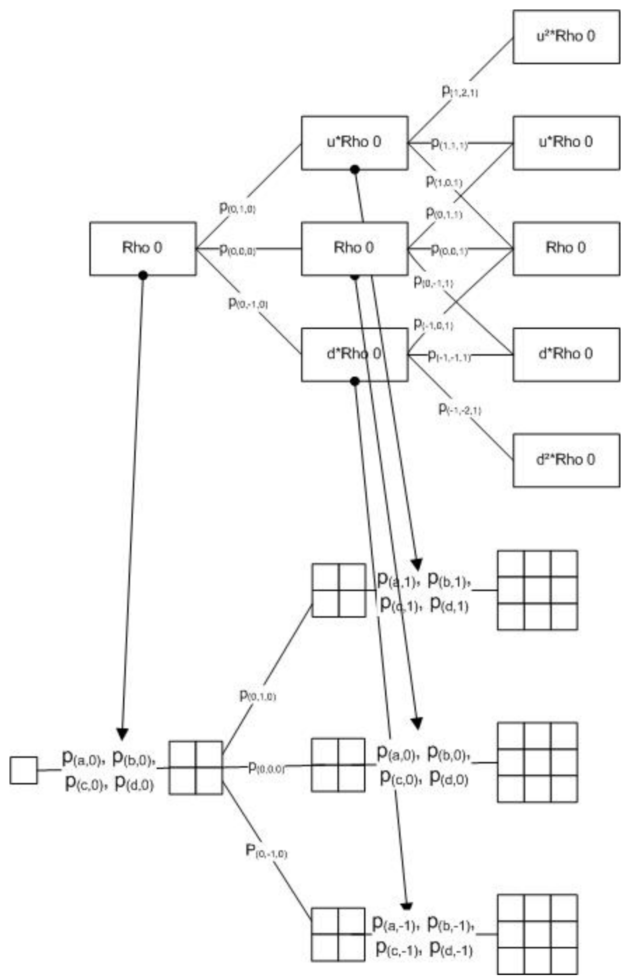

Figure 5: Structure of the combined tree

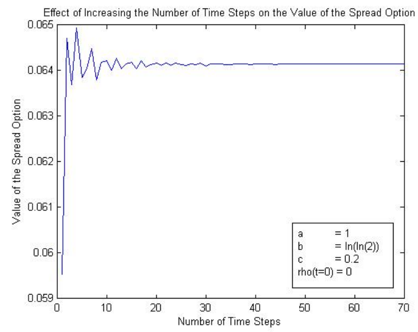

Figure 6: Effect of Varying the Number of computational Time Steps on the Value of the Spread Option in the SC-Model

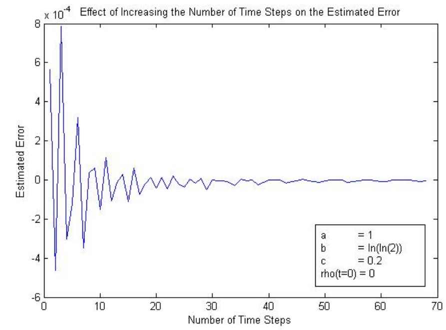

Figure 7: Effect of Varying the Number of computational Time Steps on the Estimated Errors in the SC-Model

constant Correlation Model (CC-Model) overprices the spread option and the correlation hedge parameter in the case of stochastic correlation.

6.1 Numerical Convergence

We compute the value of the option varying the number of time steps n, from $n = 1$ to 70, in the case of stochastic and constant correlation. An estimate for the error is calculated in both cases by subtracting the values found for the different time steps from the value computed with $n = 70$. Note in Figure 6 that the value of the spread option with stochastic correlation converges quickly. The corresponding estimated errors are illustrated in Figure 7. One can see that from about 30 time steps the price can be indicated with an accuracy of 4 digitals. The CC-Model does not converge considerably quicker than the SC-Model, which is shown by the estimated errors for this model in Figure 8. Subtracting the estimated errors from each other allows us to state that from 30 time steps the performance of both models considering computational convergence is equal (see Figure 9).

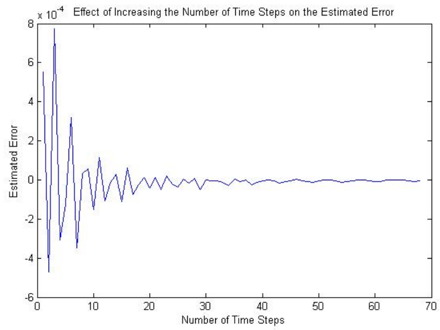

Figure 8: Effect of Varying the Number of computational Time Steps on the Estimated Errors in the CC-Model

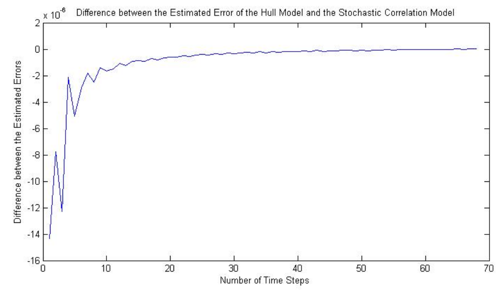

Figure 9: Difference between the Estimated Errors

6.2 Correlation Parameters

The correlation structure between the stocks affects the price of a spread option on these substantially. This can already be demonstrated for the CC-Model. In order to show the general relationship between correlation and the price of the option we analyze the effect of an increase in correlation on the price in the CC-Model. In Figure 10 the inverse relationship is illustrated: An increase in the correlation results in a lower spread option price. Furthermore, the slightly concave graph indicates that higher correlations have a bigger impact on the price.
In a next step we want to break down the influence of the parameters of y on the spread option value in the SC-Model and compare this to the CC-Model.

Sensitivity of the Price with respect to the Volatility
In the SC-Model we set the mean-reverting level as well as the value of $y(0)$ in $t = 0$ to $\ln(\ln(2))$, which is equivalent to $\rho(0) = 0$, and vary the volatility of y, i.e. c. The price of the spread option decreases with a rise in the volatility of the correlation (see Figure 11). Since higher correlations have a bigger impact on the price, as we have seen before, an increase of the volatility of the mean reverting stochastic correlation causes a decrease of the prices of the derivative. In order to compare these results to the equivalent CC-model we set the correlation $\rho \equiv 0$. The comparison of the two graphs shows that

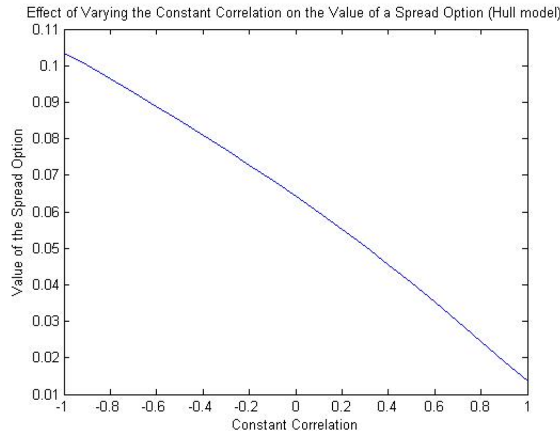

Figure 10: Effect of Varying Correlation on the Value of a Spread Option

the CC-Model systematically overestimates the price of the option as the volatility of the correlation increases (see Figure 11).

Sensitivity of the Price with respect to the Mean-reverting Level
Analyzing the impact of the mean-reverting level on the price of the spread option we find a similar effect. We vary b from $\ln(\ln(\frac{4}{3}))$, which corresponds to a mean-reverting level for $\rho = -0.5$, to $\ln(\ln(4))$ (equivalent to a mean-reverting level for $\rho = 0.5$) and set $\rho(0)$ to the respective mean-reverting level value. All other parameters are left constant. Figure 12 shows a negative interrelation between the values of the mean-reverting level and the values of the spread options. The higher the long-term mean of the correlation the less probable become big spreads between the two shares and therefore the value of the spread option has to fall with an increase in the mean-reverting level.

In order to compare the results of the SC-Model to the CC-Model we compute the values of the CC-Model assuming that the constant correlation ρ is set to the mean-reverting level for ρ_t in the SC-Model. Figure 12 visualizes that the CC-Model overstates the option values for negative long-term mean values the most, i.e. higher and lower correlations than the mean-reverting level are possible. As the lowering effect of the highly positive correlations is bigger (see Figure 10) the prices of the SC-Model are lower than those of the CC-Model. This effect is, of course, not as distinct for very positive mean-reverting levels.

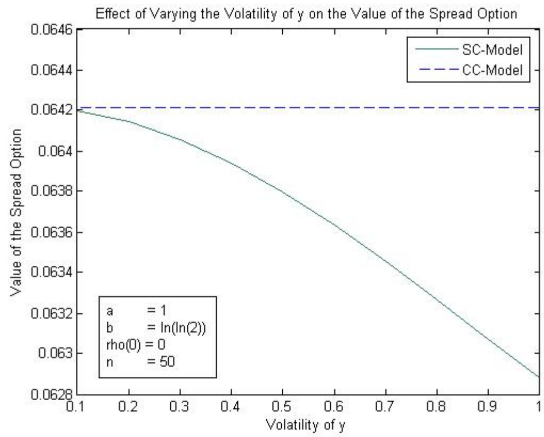

Figure 11: Effect of Varying the Volatility of y_t on the Value of the Spread Option

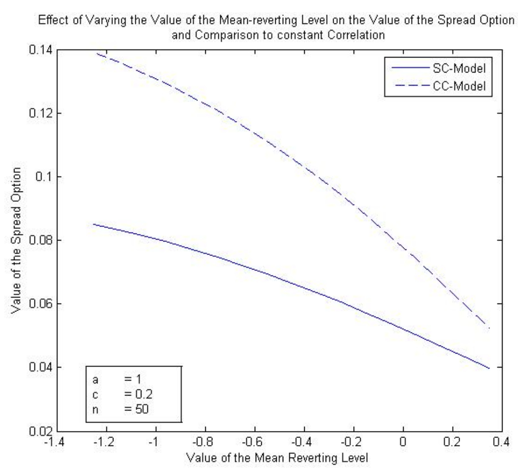

Figure 12: Effect of Varying the Value of the Mean-reverting Level on the Value of the Spread Option

6.3 Hedging Parameter

For the basic scenario we want to compute the hedging parameters $\Delta_i = \frac{\delta V}{\delta S_i(0)}$, $\nu_i = \frac{\delta V}{\delta \sigma_i}$ and $\frac{\delta V}{\delta \rho}$, $i \in \{1,\ 2\}$, in $t = 0$, where V is the value of the spread option, which is among others dependent on S_i, σ_i and ρ_t. $V(S_i(0))$ $(V(\sigma), V(\rho))$ we denote the value of the option varying $S_i(0)$ (σ_i and ρ repectively). In Table 2 we provide these hedge ratios for sample values of S_1 in $t = 0$.

Table 2: Hedge Ratios for chosen values

Hedging Parameter	In the Money $S_1 = 1.6$	At the Money $S_1 = 1$	Out of the Money $S_1 = 0.4$
Δ_1	0.97944	0.62217	0.012024
Δ_2	-0.93069	-0.37846	-0.00088247
ν_1	0.14303	0.3677	0.011437
ν_2	0.066843	0.19963	0.0043561
$\frac{\delta V}{\delta \rho}$	-0.0076071	-0.030068	-0.00030971

Delta Hedging Parameters
We compute the Δ sensitivity of the spread option by locally altering the value of S_1, S_2 in $t = 0$ respectively, i.e.

$$\text{Delta Hedge Ratio} = \frac{V(S_i(0) + h) - V(S_i(0))}{h}.$$

Where h is a small number. Figure 13 shows that the hedging parameter rises exponentially with an increase in S_1 for out of the money options and with a decreasing radient for in the money options and thus, it corresponds to the Δ sensitivity of a plain vanilla call option. Figure 14 exhibits the exact opposite behavior: It falls exponentially for out of the money options and with a decreasing radient for in the money options. This appearance is due to the payoff structure of the spread option in T. Not surprisingly, the results for Δ_i are exactly the same in the constant correlation case as the derivative $\frac{\delta V}{\delta S_i(0)}$ is not influenced by the correlation structure.

Vega Hedging Parameters
We compute the Vega hedging parameter of the spread option by locally altering the value of σ_1, σ_2 in $t = 0$ respectively, i.e.

$$\text{Vega Hedge Ratio} = \frac{V(\sigma_i + h) - V(\sigma_i)}{h}.$$

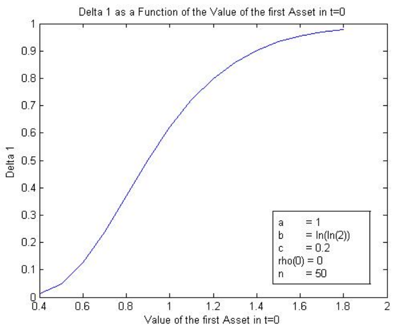

Figure 13: Δ_1 as a Function of the Value of S_1 in t_0

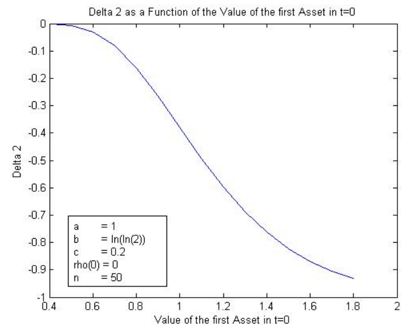

Figure 14: Δ_2 as a Function of the Value of S_1 in t_0

Note that the sensitivity of the spread option with respect to the volatility of the underlying assets increases for out of the money options approaching the strike price, the sensitivity is the highest at the money and decreases in the money (see Figures 15 and 16). These results are in accordance with plain vanilla options, where the influence of the volatility is the biggest for options close at the money and at the money. As before the findings do not differ from those we get for the Vega hedging parameter using the CC-Model because we model the correlation independent from the variance structure of the underlying assets.

Correlation Hedging Parameters

We approximate the correlation hedging parameter $\frac{\delta V}{\delta \rho}$ by

$$\frac{V(\rho(0)+h)-V(\rho(0))}{h},$$

i.e. we alter the correlation in $t=0$, leaving the long-term mean constant. Figure 17 reflects the negative relationship between correlation and the value of the spread option that we have already pointed out earlier. The hedging parameter falls for out of the money options and increases in the money. The sensitivity with respect to the correlation is the highest at the money. To compute the respective hedging parameter for the CC-Model we variate the constant correlation. The hedge parameter computed in the CC-Model exhibits similar features (see Figure 17) as in the SC-Model. However, the CC-Model overestimates the sensitivity with respect to changes in the underlying correlation structure as it does not take into account the long-term

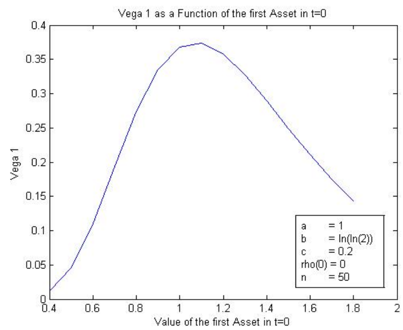

Figure 15: ν_1 as a Function of the Value of S_1 in t_0

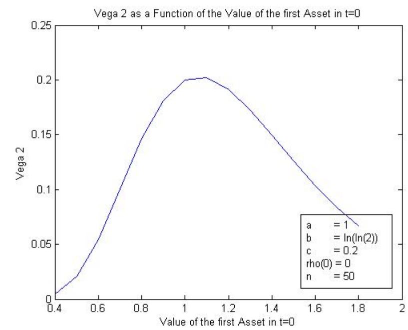

Figure 16: ν_2 as a Function of the Value of S_1 in t_0

mean. The correlation can be hedged with another instrument involving correlation or with another spread option on the same stocks but with different maturities.

7 Summary and Conclusion

We have developed and implemented a tree model to price spread options on underlyings which are stochastically correlated based on a system of stochastic processes with a mean-reverting process for the stochastic correlation. This model relaxes the constant correlation assumption in the existing literature. The tree model converges quickly and the value of the spread option can be indicated with four digitals computing more than 30 time steps. Thus, the convergence of the stochastic correlation model is as fast as the constant correlation tree model proposed by Hull (2002). Our framework allows us to examine several effects of a mean-reverting stochastic correlation. We show that the equivalent constant correlation model overestimates the value of a spread option as well as the hedging parameter for a correlation hedge.

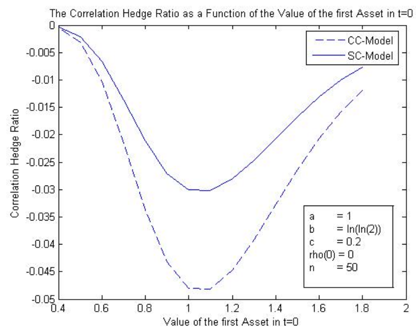

Figure 17: The Hedging Parameter of the Correlation as a Function of the Value of S_1 in t_0

A Proof of Proposition 1

Basic Equations for the bidimensional binomial approximation:

$$p_1 S_1 u_1 + (1-p_1) S_1 d_1 = S e^{r\Delta t} \tag{26}$$

$$p_2 S_2 u_2 + (1-p_2) S_2 d_2 = S e^{r\Delta t} \tag{27}$$

$$(u_1-1)^2 p_1 + (1-p_1)(d_1-1)^2 - (p_1(u_1-1) + (1-p_1)(d_1-1))^2 = \sigma_1^2 \Delta t \tag{28}$$

$$(u_2-1)^2 p_2 + (1-p_2)(d_2-1)^2 - (p_2(u_2-1) + (1-p_2)(d_2-1))^2 = \sigma_2^2 \Delta t \tag{29}$$

$$\begin{aligned}(u_1-1)(u_2-1)p_a + (u_1-1)(d_2-1)p_b \\ +(d_1-1)(u_2-1)p_c + (d_1-1)(d_2-1)p_d \\ -(u_1-1)p_1 + (d_1-1)(1-p_1)(u_2-1)p_2 \\ +(d_2-1)(1-p_2) &= \sigma_1\sigma_2\rho\Delta t\end{aligned} \tag{30}$$

$$p_a + p_b + p_c + p_d = 1 \tag{31}$$

$$p_a + p_b = p_1 \tag{32}$$

$$p_d + p_c = 1 - p_1 \tag{33}$$

$$p_b + p_d = 1 - p_2 \tag{34}$$

$$p_a + p_c = p_2 \tag{35}$$

- Equation (26) and Equation (27): The expectation of S_t in the tree has to meet the expectation of S_t in continuous time: $E[S_t] = S(0)e^{rt}$. The approximation is exact in this case.

- Equation (28) and Equation (29): $Var[S_t] = S_0^2 e^{2rt}(e^{\sigma^2 t} - 1)$. However, for reasons of simplification we use the fact that $Var[\frac{dS}{S}] = \sigma^2 dt$, which implies[4]

$$Var[\frac{\Delta S}{S}] = \sigma^2 \Delta t$$

- Equation (30): The same simplification is used for the covariance and the correlation respectively.
- Equation (31): The probabilities of the four branches have to sum up to 1.
- Equations (32), (33), (34), (35) are derived from the marginal probabilities of a single asset.

B Proof of Proposition 2

Reformulate Equation (16)

$$\begin{aligned} & u_1u_2(p_a - p_1p_2) + u_1d_2(p_b - p_1(1-p_2)) \\ & +d_1u_2(p_c - (1-p_1)p_2)+ \\ & d_1d_2(p_d - (1-p_1)(1-p_2)) - \sigma_1\sigma_2\rho\Delta t \quad = \quad 0 \end{aligned} \tag{36}$$

From Equations (7) to (11) we get:

$$p_a = p_1 - p_b \tag{37}$$
$$p_b \quad \text{free} \tag{38}$$
$$p_c = (1-p_1) - p_d = -p_1 + p_2 + p_b \tag{39}$$
$$p_d = (1-p_2) - p_b. \tag{40}$$

Substituting these expressions in Equation (36) and solving for p_b leads to

$$p_b = p_1(1-p_2) + \frac{\sigma_1\sigma_2\rho\Delta t}{(d_2-u_2)(u_1-d_1)}. \tag{41}$$

[4]This simplification has already been used by Hull (2002).

Substituting Equation (41) in Equations (37, 39, 40) we get

$$\begin{aligned} p_a &= p_1p_2 - \frac{\sigma_1\sigma_2\rho\Delta t}{(d_2-u_2)(u_1-d_1)} \\ p_c &= p_2(1-p_1) + \frac{\sigma_1\sigma_2\rho\Delta t}{(d_2-u_2)(u_1-d_1)} \\ p_d &= (1-p_1)(1-p_2) - \frac{\sigma_1\sigma_2\rho\Delta t}{(d_2-u_2)(u_1-d_1)} \end{aligned} \tag{42}$$

It follows from Equation (12) for u_1, d_1 and for u_2 and d_2 respectively:

$$u_1 - d_1 = \frac{e^{r\Delta t}-d_1}{p_1}, \quad u_2 - d_2 = \frac{e^{r\Delta t}-d_2}{p_2} \tag{43}$$

Substituting Equation (43) in Equation (42) we obtain

$$p_a = p_1p_2 - \frac{\sigma_1\sigma_2\Delta t p_1p_2\rho}{(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})} \tag{44}$$

$$p_b = p_1(1-p_2) + \frac{\sigma_1\sigma_2\Delta t p_1p_2\rho}{(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})} \tag{45}$$

$$p_c = p_2(1-p_1) + \frac{\sigma_1\sigma_2\Delta t p_1p_2\rho}{(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})} \tag{46}$$

$$p_d = (1-p_1)(1-p_2) - \frac{\sigma_1\sigma_2\Delta t p_1p_2\rho}{(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})} \tag{47}$$

Hence, ρ is restricted by the conditions for the probabilities ($0 \le p_i \le 1$):

p_a :

$$\frac{(p_1p_2-1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2p_1p_2\Delta t} \le \rho \le \frac{(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2\Delta t}$$

p_b :

$$\frac{p_1(p_2-1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2p_1p_2\Delta t} \le \rho \le \frac{(1-p_1(1-p_2))(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{p_1p_2\sigma_1\sigma_2\Delta t}$$

p_c :

$$\frac{p_2(1-p_1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2p_1p_2\Delta t} \le \rho \le \frac{(1-p_2(1-p_1))(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{p_1p_2\sigma_1\sigma_2\Delta t}$$

p_d :

$$\frac{((1-p_1)(1-p_2)-1)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{\sigma_1\sigma_2p_1p_2\Delta t} \le \rho \le \frac{(1-p_1)(1-p_2)(e^{r\Delta t}-d_1)(d_2-e^{r\Delta t})}{p_1p_2\sigma_1\sigma_2\Delta t}$$

C General Trinomial Tree Probabilities

In the trinomial tree the probabilities of $y(l,t)$ moving to $y(\kappa-1,t), y(\kappa,t)$ and $y(\kappa+1,t)$ are chosen to match the first and second moments of the three point-jump process of the change in $y(l,t)$ to the continuous distribution. Thus, the following equations must be satisfied:

$$\begin{aligned} p_{l,\kappa-1}(\kappa-1-l)\Delta y + p_{l,\kappa}(\kappa-l)\Delta y + p_{l,\kappa+1}(\kappa+1-l)\Delta y &= \mu\Delta t \quad (48) \\ p_{l,\kappa-1}(\kappa-1-l)^2\Delta y^2 + p_{l,\kappa}(\kappa-l)^2\Delta y^2 & \\ +p_{l,\kappa+1}(\kappa+1-l)^2\Delta y^2 - (\mu\Delta t)^2 &= c^2\Delta t \quad (49) \\ p_{l,\kappa-1} + p_{l,\kappa} + p_{l,\kappa-1} &= 1, \quad (50) \end{aligned}$$

where

$$\mu = a(b-y).$$

It follows from (50)

$$p_{l,\kappa} = 1 - p_{l,\kappa-1} - p_{l,\kappa+1}. \tag{51}$$

Substituting Equation (51) in (48) and reformulating it we get $(\mu(l,t) := \mu)$

$$-p_{l,\kappa-1}\Delta y + (\kappa-l)\Delta y + p_{l,\kappa+1}\Delta y = \mu\Delta t, \tag{52}$$

which is equivalent to

$$p_{l,\kappa+1} = \frac{\mu\Delta t}{\Delta y} - (\kappa-l) + p_{l,\kappa-1} \tag{53}$$

Substituting Equations (51) and (53) in (49) we have

$$\begin{aligned} & p_{l,\kappa-1}(\kappa-1-l)^2(\Delta y)^2 + \\ & (1 - p_{l,\kappa-1} - \tfrac{\mu\Delta t}{\Delta y} + (\kappa-l) - p_{l,\kappa-1})(\kappa-l)^2(\Delta y)^2 \\ & +(\tfrac{\mu\Delta t}{\Delta y} - (\kappa-l) + p_{l,\kappa-1})(\kappa+1-l)^2(\Delta y)^2 \quad = \quad \mu^2(\Delta t)^2 + c^2\Delta t, \end{aligned}$$

which is equivalent to

$$\begin{aligned} p_{l,\kappa-1} &= \frac{c^2\Delta t}{2(\Delta y)^2} + \frac{\mu^2(\Delta t)^2 + 2(l-\kappa)\Delta y\mu\Delta t + (l-\kappa)^2(\Delta y)^2}{2(\Delta y)^2} \\ &\quad -\frac{\mu\Delta t + (l-\kappa)\Delta y}{2\Delta y} \end{aligned} \tag{54}$$

With $\eta = \mu\Delta t + (l-\kappa)\Delta y$

$$p_{l,\kappa-1} = \frac{c^2\Delta t}{2(\Delta y)^2} + \frac{\eta^2}{2(\Delta y)^2} - \frac{\eta}{2\Delta y}.$$

Substituting Equation (54) in (53) we get

$$p_{l,\kappa+1} = \frac{c^2\Delta t}{2(\Delta y)^2} + \frac{\eta^2}{2(\Delta y)^2} + \frac{\eta}{2\Delta y} \tag{55}$$

Substituting Equations (54) and (55) in (51) we obtain

$$p_{l,\kappa} = 1 - \frac{c^2\Delta t}{(\Delta y)^2} - \frac{\eta^2}{(\Delta y)^2} \tag{56}$$

D Specific Choice of Tree Probabilities

As Hull and White do not provide the proof for this dynamic rule in their paper the restrictions are shown for the example $\kappa = l$.
In the case of $\kappa = l$ the probabilities are

$$p_{l,l+1} = \frac{c^2\Delta t}{2(\Delta y)^2} + \frac{\mu^2(\Delta t)^2}{2(\Delta y)^2} + \frac{\mu\Delta t}{2\Delta y} \tag{57}$$

$$p_{l,l} = 1 - \frac{c^2\Delta t}{(\Delta y)^2} - \frac{\mu^2(\Delta t)^2}{(\Delta y)^2} \tag{58}$$

$$p_{l,l-1} = \frac{c^2\Delta t}{2(\Delta y)^2} + \frac{\mu^2(\Delta t)^2}{2(\Delta y)^2} - \frac{\mu\Delta t}{2\Delta y} \tag{59}$$

To obtain positive probabilities, which are smaller than 1 we have to ensure that Equations (57) to (59) are positive:

$$p_{l,l+1} = \tfrac{c^2\Delta t}{2(\Delta y)^2} + \tfrac{\mu^2(\Delta t)^2}{2(\Delta y)^2} + \tfrac{\mu\Delta t}{2\Delta y} \geq 0 \quad \Leftrightarrow \quad \tfrac{\mu^2(\Delta t)^2 + \mu\Delta t\Delta y}{c^2\Delta t} \geq -1$$

Substituting $c^2\Delta t = \frac{1}{3}(\Delta y)^2$ results in:

$$\tfrac{\mu^2(\Delta t)^2 + \mu\Delta t\Delta y}{(\Delta y)^2} \geq -\tfrac{1}{3} \quad \Leftrightarrow \quad (\tfrac{\mu\Delta t}{\Delta y} + \tfrac{1}{2})^2 + \tfrac{1}{12} \geq 0,$$

which does not impose any constraints on the parameters.

$$\begin{array}{lll} p_{l,l} = 1 - \frac{c^2\Delta t}{(\Delta y)^2} - \frac{\mu^2(\Delta t)^2}{(\Delta y)^2} \geq 0 & \Leftrightarrow & -\sqrt{\frac{2}{3}} \leq \frac{\mu\Delta t}{\Delta y} \leq \sqrt{\frac{2}{3}} \\ p_{l,l-1} = \frac{c^2\Delta t}{2(\Delta y)^2} + \frac{\mu^2(\Delta t)^2}{2(\Delta y)^2} - \frac{\mu\Delta t}{2\Delta y} \geq 0 & \Leftrightarrow & -(\frac{\mu\Delta t}{\Delta y} - \frac{1}{2})^2 - \frac{1}{12} \leq 0, \end{array}$$

which does not impose any constraints on the parameters.
Thus,

$$-\sqrt{\frac{2}{3}} \leq \frac{\mu\Delta t}{\Delta y} \leq \sqrt{\frac{2}{3}}.$$

Equivalently, for $k = l + 1$

$$1 - \sqrt{\frac{2}{3}} \leq \frac{\mu\Delta t}{\Delta y} \leq \sqrt{\frac{2}{3}}$$

and for $k = l - 1$

$$-\sqrt{\frac{2}{3}} \leq \frac{\mu\Delta t}{\Delta y} \leq -1 + \sqrt{\frac{2}{3}}.$$

Considering all three branching methods the following dynamic rules for the choice of the parameter k can be derived:

$$k = \begin{cases} l+1 & \text{if} \quad \frac{\mu\Delta t}{\Delta y} \geq \sqrt{\frac{2}{3}} \\ l & \text{if} \quad -\sqrt{\frac{2}{3}} < \frac{\mu\Delta t}{\Delta y} < \sqrt{\frac{2}{3}} \\ l-1 & \text{if} \quad \frac{\mu\Delta t}{\Delta y} \leq -\sqrt{\frac{2}{3}} \end{cases} \tag{60}$$

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