Optimal portfolio allocation with Asian hedge funds and Asian REITs

Stephan Höcht
HVB-Institute for Mathematical Finance
Technische Universität München, Germany
E-mail: hoecht@ma.tum.de

Kah Hwa Ng
Director, Risk Management Institute
National University of Singapore
E-mail: rmingkh@nus.edu.sg

Jürgen Wolf
Technische Universität München, Germany and
National University of Singapore
E-mail: wolfjuer@gmail.com

Rudi Zagst
Director, HVB-Institute for Mathematical Finance
Technische Universität München, Germany
E-mail: zagst@ma.tum.de

Abstract: During the past years, the institutional interest in investments into hedge funds and real estate investment trusts has grown considerably. In this paper the benefits of investing in these asset classes are analyzed by applying models that recognize higher-order moments or the whole return distribution like the power-utility, Omega, and Score-value model. Trying to obtain more general results than those we can find from historical data only, we modelled the asset returns by Markov switching processes and did a Monte Carlo study. Within this design we analyzed the optimal allocations to hedge funds and REITs statically and with monthly reallocations based on data from Asian markets. Our main findings are that in the static case the utility model and the Score model are dominant, whereas the mean-variance model appears to be the model of first choice in the dynamic case. In both settings hedge funds are the most dominant asset of the optimal portfolios. REITs are mainly used for diversification and added at comparably lower rates.

Keywords: Alternative Investments, Asset Allocation, Higher-Order Moments, Markov-Switching Autoregressive Model

1 INTRODUCTION

The optimal allocation of institutional investors’ capital to various asset classes is often referred to as inter-asset or mixed-asset diversification. Investors have been searching for possibilities to enhance portfolio performance beside the traditional asset classes bonds and stocks. It is obvious that investments which depend on different factors driving the returns are likely to provide for the best diversification potential. Thus, many investors seeking diversification looked for stocks and bonds in other markets, i.e. internationalized the portfolio. Unfortunately, in times of general downturn on bond and stock markets or big crisis as the Asian crisis 1997, the Russian debt crisis 1998, the burst of the internet bubble 2000, and the attack on the World Trade Center in New York 2001, diversification is most needed by the investor but very difficult to obtain within these traditional asset classes – even if the portfolio is diversified internationally. Bertero and Mayer (1990), King and Wadhwani (1990), and King et al. (1994) analyzed the period surrounding the crash of 1987 and found greater integration of the world’s stock markets. What is most needed by institutional investors at these times are assets with low correlation with respect to bond and stock markets. This explains the growing institutional interest in investments into hedge funds and real estate during the last years.

If risk is defined as standard deviation, it has been demonstrated that the inclusion of hedge funds promises risk reduction without loss of expected return. But as Scott and Horvath (1980) have shown, rational investors have also clear preferences for higher order moments.
Unfortunately, including hedge funds and real estate investment trusts (REITs) into portfolios very often leads to lower skewness and higher excess kurtosis (see, e.g., Brooks and Kat (2002) or Brunel (2004)) which is the exact opposite of what investors prefer. Therefore, relying on the mean and the standard deviation only is dangerous. Most of the studies investigating gains in portfolio performance when alternative investments are included carry out the analysis by introducing the alternative assets into a portfolio of traditional bonds and equity. Ziobrowski and Ziobrowski (1997), for example, added US real estate to bonds and stocks and analyzed the risk reduction in the optimal mean-variance portfolios. They found that real estate risk had been grossly underestimated and that for investors with a low-risk tolerance the higher levels of real estate risk can eliminate all real estate diversification benefits. Lee and Stevenson (2005) used real estate investment trusts to add to the portfolio of stocks and bonds. Their analysis shows that REITs’ attractiveness as a diversification asset increases with the length of the holding period. In addition, REITs provide for return enhancement properties at the lower end of the efficient frontier and for risk reduction at the top end of the efficient frontier. Conover et al. (2002) confirm this by showing that foreign real estate has a significant weight in efficient international portfolios.

Studies, concerning hedge funds in the mixed-asset portfolio are numerous as well: Martellini and Vaisis (2006), e.g., analyzed the return enhancement benefits and risk reduction benefits of adding different hedge fund styles into existing portfolios. McFall Lamm (2003) as well as Brunel (2004) and Brunner and Hafner (2006) find that hedge funds have the potential to improve the mean-variance properties of a portfolio but create very unattractive higher moments. The majority of empirical studies we found in the literature concentrate on analyzing the effects of including just one alternative asset class in a mixed-assets portfolio. In this study we want to concentrate on the Asian markets and find the special implications of adding Asian alternative assets to the portfolio of stocks and bonds. Therefore, we choose an Asian hedge funds index and an Asian REITs index which are added to the portfolio at the same time. Furthermore, we analyze the differences between a static optimization like in Brunner and Hafner (2006) and a dynamic portfolio optimization as performed by, e.g., Grauer and Hakansson (1987). The goal of this study is to examine the optimal portfolio fractions of alternative investments (hedge funds and REITs). We look at three different investor types with each having three different investment horizons: 1 year, 3 years, and 5 years. We distinguish one very conservative investor A, one moderately risk averse investor B and one more aggressive investor C.

As mentioned above, we start with a static optimization like in Brunner and Hafner (2006), i.e. we look at the investment horizon as one single period. Moreover, we also run a dynamic optimization as introduced by, e.g., Grauer and Hakansson (1987). In this part, the investors are allowed to rearrange the portfolio weights at the end of each month to respond to market developments. We directly compare the allocation results with the results of the statically optimized allocations. Eventually, we analyze the sensitivity of the portfolio weights to biases of the hedge funds data. We accomplish this by correcting for the survivorship and backfill biases and rerunning the portfolio optimization.

The remainder of this paper is organized as follows: In Section 2 we examine the sample of Asian asset returns and establish their main characteristics. Section 3 covers the model used for the return processes and Section 4 introduces the technique applied for the simulation of the return paths. The investor types we look at are defined in Section 5. Section 6 describes the results of the static optimization, Section 7 the results of the dynamic optimization. We provide for a sensitivity analysis with respect to the input parameters in Section 8 and conclude in Section 9.

2 THE ASIAN DATA SET

We use indices to proxy for log returns of stocks and bonds from Asian countries. The MSCI All Country Asia Price Index is used for investments into stocks (in the following referred to as Stocks) while Asian bond returns are represented through the J.P. Morgan Emerging Markets Bond Price Index (from now on referred to as Bonds). As currency risk between different Asian countries can be eliminated (for example through currency swaps) the portfolio structuring is looked at separately and for convenience both indices are used in their U.S. Dollar representation. The sample data of monthly log returns covers the time period from January 2000 until July 2006, i.e. 79 returns (three returns were taken out due to the fact that the very small sample showed extremely atypical behaviour. These are the months April 2000, July 2000 and September 2001). Obviously, this sample has very uncommon and therefore unexpected statistical properties: stocks show a very low mean of 0.20% per month (2.40% p.a.) compared to a mean of Bonds which is more than four times higher (0.85% per month or 10.20% p.a.)

In this study, hedge fund returns are represented by the Eurekahedge Asian Hedge Fund Index which is composed of hedge funds of all styles and across all Asian countries. REITs’ returns are proxied by the EPRA/NAREIT Asia Total Return Index which also comprises different styles and regions.

We observe that the EPRA/NAREIT Asia Total Return Index (from now on referred to as REITs) has a much higher standard deviation as the Eurekahedge Asian Hedge Fund Index (from now on just Hedge Funds). Especially REITs show more than normal highly negative returns and more returns near the mean. Both properties indicate positive excess kurtosis. Surprisingly, Hedge Fund returns have no extreme negative values at all but tend to show more than normal positive values. We will analyze these impressions in more detail by scrutinizing the sample data set: the summary of the statistics of all return distributions is shown in Table 1.
Hedge funds data of world indices is supposed to have higher mean returns than equity (see, e.g., Brunel (2004)). The negative skewness of Hedge Funds returns is in line with our expectations even if it is very small in absolute terms (-0.01). On the other hand, the high negative value of Hedge Funds’ excess kurtosis (-0.68) is very surprising. Brooks and Kat (2002) amongst others showed that in general the low standard deviation and high mean return of hedge funds come at the price of unfavorable low skewness and high excess kurtosis. Contrary, the observed in-sample excess kurtosis of -0.68 is very welcomed by investors.

The above mentioned typical positive autocorrelation found in hedge fund data is confirmed in this sample by applying the Ljung-Box-Test (see, e.g., Box and Ljung (1978)). Hedge Funds’ autocorrelation coefficient of lag 1 is 0.27 and therefore high enough to exceed the critical value of the 5% significance level. This confirms the results of other studies, e.g. Brooks and Kat (2002) who found evidence of highly significant positive first-order autocorrelation in hedge fund data. Probable explanations for the autocorrelation in hedge fund returns are illiquidity exposure, i.e. the problem of unavailable market prices of investments, and performance smoothing, i.e. intentionally reported low volatile returns. We also tested for effects of volatility clustering by computing the Ljung-Box-statistic Q(3) for squared returns. Volatility clustering describes the phenomenon that large changes in prices tend to follow large changes while small changes tend to follow small changes. We can reject the null hypothesis for no autocorrelation in squared residuals for Hedge Funds only. That gives hint for volatility clusters in the Hedge Funds data.

Altogether, the Hedge Funds data in the historical sample shows unexpectedly low mean return and surprisingly attractive excess kurtosis. In comparison to the traditional asset classes, the levels of standard deviation and skewness confirm the findings of other studies about world indices of hedge funds. The significant positive autocorrelation properties also are in line with other studies and indicate the above explained illiquidity exposure or intentional performance smoothing (see, e.g., Brunner and Hafner (2006)).

Lee and Stevenson (2005) and McFall Lamm (2003) have analyzed the benefits of adding REITs into a portfolio of traditional assets. They found that REITs returns typically have a mean and a standard deviation between that of bonds and stocks. In the historical sample of this study the returns of Stocks are very low and Bonds have a very high mean which actually lies even above that of REITs. The negative skewness (-0.74) and positive excess kurtosis (0.86) are consistent with the results of Ziobrowski and Ziobrowski (1997). Applying the Jarque-Bera- (see, e.g., Bera and Jarque (1980)) and Ljung-Box-Test, respectively, shows significant non-normality and autocorrelation present in the REITs data. Similar to the Hedge Funds returns the autocorrelation for lag 1 is very high and positive (0.21) indicating some sort of illiquidity exposure. Both autocorrelations of Hedge Funds and REITs will be taken account for with the type of Markov switching models we applied for the simulation of the return series.

If we compare the returns of the alternative investments in the sample with those of traditional Bonds, they do not look very superior. This is due to the very untypical sample distribution of Bonds. For later analysis we correct the mean return levels of all assets to weaken the problems caused by these rather unusual specifics. Those rather strange properties might be caused by the crisis on the stock market which negatively affects many hedge fund strategies as well. Therefore the hedge fund index used as proxy for investments in Asian hedge funds might also have suffered more than usual.

Introduced into a portfolio of Bonds and Stocks, the alternative assets still have the potential to provide for diversification even if they are not adjusted at all. This is shown by setting up equally weighted portfolios. Table 2 contains the descriptive statistics of the portfolios with different fractions of alternative investments (0% in the first column means that 0% are invested in alternative investments, i.e. the portfolio contains of 50% Bonds and 50% Stocks. 20% means that 20% are invested in alternative investments (10% in Hedge Funds and 10% in REITs), the remaining 80% are invested in traditional asset classes (40% in Bonds and 40% in Stocks), etc.). Introducing alternative investments into the traditional portfolio seems very promising in a mean-variance world but tends to render the values for the third and fourth moment in an unfavourable manner. The return series of alternative investments are intentionally created to have low or even negative correlation to traditional asset classes to provide higher diversification. Therefore the performance measured by the Sharpe Ratio can be improved by adding Hedge Funds and REITs to the starting portfolio.
The traditional mean-variance optimization does not realize the negative attributes associated with skewness and excess kurtosis of alternative investments’ return distributions. When the adjusted Sharpe Ratio is applied this becomes obvious: the unattractive features of alternative assets negatively influence the performance. As was shown in Brunel (2004) alternative assets’ high weights in the optimal mean-variance portfolios might just be a way of paying for undesired negative skewness and positive excess kurtosis.

### 3 PORTFOLIO OPTIMIZATION MODELS

We will compare four different optimization models. The first one is the traditional mean-variance portfolio optimization which maximizes the mean-variance tradeoff of the investor. It is defined by

$$\max_w w^T \mu - \lambda \frac{1}{2} w^T \Sigma w,$$

where $\lambda$ describes the risk aversion of the investor, $\Sigma$ denotes the covariance-matrix, and $\mu$ the vector of expected asset returns. We also apply a full investment constraint and prohibit short-selling.

Since we pointed out the importance of higher moments for the portfolio optimization, we also employ a power-utility optimization and portfolio optimizations which maximize the mean-variance tradeoff.

The power utility optimization problem is given by

$$\max_w E[U(1 + R(w))].$$

The power utility optimization problem is given by

$$\max_w \left[ \max_{\gamma} \left( \frac{1}{\gamma} (1 + R(w)) \right) \right], \quad \gamma < 1, \gamma \neq 0$$

where $\gamma$ is the risk aversion parameter and $R(w)$ the portfolio return for portfolio weights $w$.

$\Omega$ is defined as the ratio of probability weighted gains to losses in respect of the return threshold $\tau$, i.e. the upside potential divided by the downside potential (see Keating and Shadwick (2002)). The according $\Omega$-optimization problem is given by

$$\max_w \Omega (R(w)) = \max_w \frac{E[R(w) - \tau]^+}{E[\tau - R(w)]^-},$$

where $\tau$ is the loss threshold.

The formula of $\Omega$ with the risk-free rate $r_f$ used as threshold. The according Score-value optimization is given by

$$\max_w \frac{\lambda}{1} E[R(w) - \tau] - \lambda_{Sc} \cdot E[\tau - R(w)]$$

where $\lambda_{Sc}$ is the risk aversion parameter.

### 4 MODELING AND SIMULATION OF ASSET RETURNS

The simulation of the returns has to allow for the fact that the peculiarities of alternative assets’ return distributions negatively affect the investor. Amongst others, Clark (1973) has shown that the mixture of normal distributions gives the opportunity to generate different values for skewness and excess kurtosis and therefore lets us model a wider range of distributions than is obtainable through the use of a single distribution (see Timmermann (2000)). A serious drawback of these time-independent mixture models is their inability to capture the phenomena of volatility clustering present in many return data series. The general idea of Markov switching models is to assume a time-dependence of the distribution parameters but restrict the possible number of different realizations to a finite number. At the same time, it extends the idea of ARCH models in that not just the volatility is time-dependent but also the mean and other parameters. Since the pioneering work of Hamilton these models have been broadly accepted and given rise to a vast research literature (see, e.g., Hansen (1992), Kim (1994), Diebold et al. (1994), Psaradakis and Sola (1998) and Clarida et al. (2003) and many other papers). In our study, we consider portfolios made up of four assets: bonds, stocks, hedge funds and REITs. The data in the historical sample is very little. Fitting a multivariate Markov switching model is therefore very unlikely to result in stable parameters. Hence we fit univariate Markov switching models for the assets and take into account the correlation between the asset returns through the residuals used for the simulation.

We apply the Markov switching model which allows for state-independent (first lag) autoregressive dynamics as defined in Timmermann (2000) to model the return processes of the assets:

$$y_t = \mu_S + \Phi \cdot (y_{t-1} - \mu_{S, t-1}) + \sigma_S \cdot \epsilon_t,$$

where $y_t$ denotes the return at time $t$, $S_t$ the state at time $t$, $\mu_S$ the mean return in state $S_t$, $\sigma_S$ the standard deviation in state $S_t$, $\epsilon_t$ the innovation at time $t$, and $\Phi$ the autocorrelation parameter. The changes of the states $S_t$ are modeled by a Markov chain. There are two possible states: if the return process is in state 1 then $S_{t+1}=1$, if it is in state 2 then $S_{t+1}=2$. The transition probabilities of changing from state $i$ at time $t$-1 into state $j$ at time $t$ are given by

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

with $P(S_{t+1}|S_t=i)=P_{ij}$. The innovations are assumed to be independently distributed with respect to all past and future realizations of the state variable and i.i.d N(0,1).
Of course, we expect $|\Phi|<1$. If the Markov chain is ergodic, i.e. $p_{11}<1, p_{22}<1$ and $p_{11}+p_{22}>0$, then there exists a unique stationary distribution with unconditional probabilities of

$$
\pi_1 = \frac{p_{21}}{p_{12} + p_{21}}
$$

(7)

for the process being in state 1 and $\pi_2 = 1- \pi_1$ for the process being in state 2.

Since we want the moments of the simulated data to match those of the historical sample, we now estimate the unknown parameter vector $\Theta$ by equating the sample moments and the theoretical distribution moments (for brevity we omit the formulas here and refer the reader to Timmermann (2000)). The actual estimation of the values of $\Theta$ is done by numerically minimizing the squared deviation between the process moments and the according sample moments.

Historical returns do not necessarily have to be a good predictor for future returns. Moreover, since our historical data set is somewhat untypical we want to include market experts’ return forecasts for all asset classes. Altogether this information is taken into account by undertaking two steps: first we adjust the overall mean of each asset class with the expert forecasts by using the Black-Litterman model with a confidence parameter $\tau$ set equal to 95% (see Black and Litterman (1992)).

<table>
<thead>
<tr>
<th>Asset</th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$p_{12}$</th>
<th>$p_{21}$</th>
<th>$\Phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonds</td>
<td>0.0129</td>
<td>-0.0053</td>
<td>0.0099</td>
<td>0.0141</td>
<td>0.4631</td>
<td>0.5276</td>
<td>0.0093</td>
</tr>
<tr>
<td>Stocks</td>
<td>0.0745</td>
<td>-0.0156</td>
<td>2.7E-8</td>
<td>0.0299</td>
<td>0.4455</td>
<td>0.1563</td>
<td>-0.1446</td>
</tr>
<tr>
<td>HFs</td>
<td>0.0229</td>
<td>-0.0016</td>
<td>0.0083</td>
<td>0.0083</td>
<td>0.4792</td>
<td>0.4907</td>
<td>0.5950</td>
</tr>
<tr>
<td>REITs</td>
<td>0.0262</td>
<td>-0.0215</td>
<td>0.0284</td>
<td>0.0578</td>
<td>0.3568</td>
<td>0.4744</td>
<td>0.1687</td>
</tr>
</tbody>
</table>

Subsequently the BL-adjusted level of the mean return together with the statistical central moments of the historical sample are used to obtain the parameter estimates of the Markov switching processes of all assets. The resulting parameter vectors for univariate Markov switching processes are displayed in Table 4.

For each pair of assets the theoretical covariance of returns is computed dependent on the covariance of their residuals. This is accomplished by extending the proof of Proposition 2 in Timmermann (2000) for the case of two assets. Because we need to take into account that the residuals stem from different regimes (namely the regimes the corresponding processes are in at that very time), we proceed as follows: first, we derive the equation for the theoretical covariance of the assets dependent on the known parameter values of the asset processes and the unknown state-dependent correlations. The corresponding formula and a comprehensive derivation can be found in Appendix A. Then we plug all known parameter values into the derived equation to determine the values of the state-dependent correlations between the assets. We apply a technique similar to the method of moments as described above. This is how we overcome the problem of data scarcity and use multivariate behaviour nonetheless. For convenience Table 5 shows the covariance in the historical sample and the covariance which is theoretically implied by the state-dependent correlations we derived above.

We find the values to lie close to the empirical covariance matrix. The state-dependent correlations of the residuals are used to draw the residuals for all assets simultaneously dependent on the actual states of the assets. Each simulated path consists of 60 returns, i.e. 5 years, for each asset. Altogether we simulate 10,000 paths with a Monte Carlo approach.

5 DEFINITION OF INVESTOR TYPES

To compare the performance of optimal portfolios for different models, we set the risk aversion parameters in all models to values which represent the same investor in terms of risk aversion. Therefore we use a characteristic benchmark portfolio for each investor considered. We look at three different investors defined by their distinctive benchmark portfolios of Bonds and Stocks (see Table 6).

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Table 5: Theoretical (implied) covariance between assets

<table>
<thead>
<tr>
<th>Covariance in (%)</th>
<th>Bonds</th>
<th>Stocks</th>
<th>Hedge Funds</th>
<th>REITs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonds</td>
<td>0.0227 (0.0140)</td>
<td>0.0113 (0.0502)</td>
<td>0.0034 (0.0108)</td>
<td>-0.0022 (0.0207)</td>
</tr>
<tr>
<td>Stocks</td>
<td>0.0113 (0.0502)</td>
<td>0.2236 (0.0168)</td>
<td>0.0034 (0.0108)</td>
<td>0.0425 (0.0084)</td>
</tr>
<tr>
<td>Hedge Funds</td>
<td>0.0034 (0.0108)</td>
<td>0.0578 (0.0168)</td>
<td>0.2236 (0.0168)</td>
<td>0.0137 (0.1735)</td>
</tr>
<tr>
<td>REITs</td>
<td>-0.0022 (0.0207)</td>
<td>0.0137 (0.0084)</td>
<td>0.2510 (0.1735)</td>
<td>0.2510 (0.1735)</td>
</tr>
</tbody>
</table>

Table 6: Characteristic benchmark weights

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Investor A</th>
<th>Investor B</th>
<th>Investor C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_{BM}$</td>
<td>75%</td>
<td>50%</td>
<td>25%</td>
</tr>
<tr>
<td>BM Bonds</td>
<td>25%</td>
<td>50%</td>
<td>75%</td>
</tr>
</tbody>
</table>

These portfolios are used to consistently determine the risk aversion parameters and the loss thresholds for all optimization models. We do this by setting each parameter to the value which makes the optimal weights within the corresponding framework match the weights in the benchmark portfolio, i.e. we choose those values for $\lambda$, $\gamma$, $\tau$ and $\lambda_{Sc}$ which let the weights of the benchmark portfolio $W_{BM}$ appear optimal in the respective framework (see Table 7).
6 RESULTS OF STATIC OPTIMIZATION

We start with the static optimization and apply all models as introduced in Section 3 for all three investor types and all investment horizons. The optimal portfolios for this case, i.e. with no reallocations during the investment period, are reported below in Table 8.

### Table 8: Optimal portfolio weights in %

<table>
<thead>
<tr>
<th>All weights in %</th>
<th>MV</th>
<th>Ut</th>
<th>Om</th>
<th>Sc</th>
<th>MV</th>
<th>Ut</th>
<th>Om</th>
<th>Sc</th>
<th>MV</th>
<th>Ut</th>
<th>Om</th>
<th>Sc</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonds 1 year</td>
<td>0</td>
<td>0</td>
<td>27</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>23</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>23</td>
<td>0</td>
</tr>
<tr>
<td>Stocks 1 year</td>
<td>13</td>
<td>13</td>
<td>23</td>
<td>13</td>
<td>8</td>
<td>8</td>
<td>7</td>
<td>8</td>
<td>8</td>
<td>7</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>HFs 1 year</td>
<td>76</td>
<td>76</td>
<td>67</td>
<td>76</td>
<td>54</td>
<td>54</td>
<td>45</td>
<td>54</td>
<td>52</td>
<td>43</td>
<td>52</td>
<td>43</td>
</tr>
<tr>
<td>REITs 1 year</td>
<td>11</td>
<td>11</td>
<td>9</td>
<td>11</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Bonds 3 years</td>
<td>43</td>
<td>43</td>
<td>62</td>
<td>43</td>
<td>36</td>
<td>36</td>
<td>40</td>
<td>36</td>
<td>32</td>
<td>29</td>
<td>32</td>
<td>29</td>
</tr>
<tr>
<td>Stocks 3 years</td>
<td>23</td>
<td>23</td>
<td>25</td>
<td>23</td>
<td>21</td>
<td>21</td>
<td>22</td>
<td>21</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>HFs 3 years</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>REITs 3 years</td>
<td>8</td>
<td>8</td>
<td>11</td>
<td>8</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Bonds 5 years</td>
<td>14</td>
<td>14</td>
<td>42</td>
<td>14</td>
<td>11</td>
<td>11</td>
<td>6</td>
<td>11</td>
<td>6</td>
<td>11</td>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>Stocks 5 years</td>
<td>3</td>
<td>3</td>
<td>6</td>
<td>3</td>
<td>7</td>
<td>7</td>
<td>0</td>
<td>7</td>
<td>0</td>
<td>7</td>
<td>0</td>
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</tr>
<tr>
<td>HFs 5 years</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>REITs 5 years</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</tbody>
</table>

### 6.1 Portfolio Allocations

From Table 8 we find that across all investors, models and investment horizons, Hedge Funds seem to be the asset of first choice. The allocation to Hedge Funds is at least 57% (Investor A, 1 year, Ω-model) but in many cases Hedge Funds constitute the entire portfolio (e.g. Investor C, 3 years, Score-model). This confirms the results of many authors who analyzed optimal portfolios with investments into hedge funds (see, e.g., Kat (2003) or Brunner and Hafner (2006)).

Further, we can see that Bonds are hardly allocated at all. While the Ω-model puts fractions from 5% to 27% into Bonds and allocates all asset classes for all investors and all time horizons, Bonds do not appear notably in any other optimal portfolio within all other frameworks. Brunner and Hafner (2006), who studied the optimal allocations for different hedge fund styles found similar results for “equity hedge” style. The picture of REITs is not that different: if REITs are allocated at all we find rather little fractions of either 1% or 9-11% across the frameworks.

If we look for example at Investor A for the 3 year time horizon his Ω-optimal portfolio contains of 63% Hedge Funds, 23% Bonds, 7% Stocks and 7% REITs. The optimal portfolios for the other models have an investment between 91% and 96% in Hedge Funds, between 4% and 8% in Stocks and only the utility optimal portfolio has 1% of REITs.

### 6.2 Constrained Optimization

In reality allocations regularly are constrained. Therefore we apply maximum constraints (80% for Bonds and Stocks, respectively 10% for Hedge Funds, and 25% for REITs) and rerun all portfolio optimizations (see Table 9).

### Table 9: Optimal portfolio weights in % (with constraints applied)

<table>
<thead>
<tr>
<th>All weights in %</th>
<th>MV</th>
<th>Ut</th>
<th>Om</th>
<th>Sc</th>
<th>MV</th>
<th>Ut</th>
<th>Om</th>
<th>Sc</th>
<th>MV</th>
<th>Ut</th>
<th>Om</th>
<th>Sc</th>
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<tbody>
<tr>
<td>Bonds 1 year</td>
<td>43</td>
<td>43</td>
<td>44</td>
<td>43</td>
<td>44</td>
<td>44</td>
<td>44</td>
<td>44</td>
<td>44</td>
<td>44</td>
<td>44</td>
<td>44</td>
</tr>
<tr>
<td>Stocks 1 year</td>
<td>23</td>
<td>23</td>
<td>24</td>
<td>23</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>HFs 1 year</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>REITs 1 year</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Bonds 3 years</td>
<td>47</td>
<td>47</td>
<td>47</td>
<td>47</td>
<td>47</td>
<td>47</td>
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<tr>
<td>Stocks 3 years</td>
<td>23</td>
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<tr>
<td>HFs 3 years</td>
<td>10</td>
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<tr>
<td>REITs 3 years</td>
<td>8</td>
<td>8</td>
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<td>8</td>
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<td>8</td>
</tr>
<tr>
<td>Bonds 5 years</td>
<td>17</td>
<td>17</td>
<td>17</td>
<td>17</td>
<td>17</td>
<td>17</td>
<td>17</td>
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<td>17</td>
<td>17</td>
<td>17</td>
<td>17</td>
</tr>
<tr>
<td>Stocks 5 years</td>
<td>45</td>
<td>45</td>
<td>0</td>
<td>45</td>
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<td>45</td>
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<tr>
<td>HFs 5 years</td>
<td>10</td>
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<td>10</td>
</tr>
<tr>
<td>REITs 5 years</td>
<td>5</td>
<td>5</td>
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</tbody>
</table>

The differences in the optimal portfolios amongst the investor types do not appear to be very big. This, of course, is at least in part a result of the values of the risk aversion parameters. More distinct results may be obtained by using externally defined benchmark returns and risk aversions which differ to a greater extent. Hagelin and Pramborg (2005) did so and applied γ-values in the range of [-60; 1] in the utility framework which resulted in more distinct allocations. In such a case the comparison of different models is very difficult as it is hardly possible to consistently parameterize them. The approach of using benchmark portfolios as applied here guarantees this consistency.

With the application of the constraints the allocations change essentially. All optimal portfolios put 10% into Hedge Funds (the maximum admissible fraction) leaving a fair fraction to redistribute as all unconstrained portfolios...
assigned higher weights to Hedge Funds before. According to the individual risk aversions, i.e. dependent on the investor type, the optimization models prefer distinct assets: similar to the unconstrained optimization, the \( \Omega \)-framework allocates differently than the other three frameworks. It increases the fraction of REITs just slightly compared to the unconstrained weights (REITs are between 10% for Investor A and a time horizon of 5 years and 23% for Investor C on the 1 year time horizon). In all portfolios a fraction similar to that of REITs is invested into Stocks. For lower levels of risk aversion these fractions increase and the \( \Omega \)-model invests more into the riskier assets REITs and Stocks which promise a higher mean return. Clearly, the most preferred asset class is Bonds. In the \( \Omega \)-optimization we are looking for returns that exceed the loss threshold \( \tau \).

Investor A imposes the lowest value of 4.06% p.a. (for investments of 5 years) for the loss threshold and Investor C the highest of 5.47% p.a. (for 1 year). Although Bonds are the asset class which promise the lowest mean return for each investment horizon of approx. 5.4% p.a. we see that this return is very close to the loss thresholds. This mean return comes together with the lowest volatility of all assets. Therefore, a portfolio with a high fraction of Bonds earns the required return at a very low level of risk causing the downside to be very small and the \( \Omega \)-measure to be very high. With an investment of 10% into Hedge Funds the model bets on high returns. The optimal weights in the other optimization frameworks behave slightly different: again, Hedge Funds are assigned the maximum possible fraction of 10%. After allocating the admissible 25% into REITs (just investor A stays slightly below 25%) the remainder is split amongst Bonds and Stocks (in fact, these three models (mean-variance-, utility- and Score model) seek to allocate higher fractions of the portfolio into REITs when the maximum constraint of 25% is loosened). While the return-seeking investor type C (lowest values of risk aversion) does not allocate Bonds at all, investors A and B are more attracted by the less risky Bonds (in terms of standard deviation). The most conservative investor A puts even more money into the less risky Bonds (42%-49%) than into Stocks (23%-35%). This is also shown by plotting the optimal portfolio weights along the mean-variance efficient frontier for the unconstrained and the constrained case (see Figure 1).

Almost the entire fraction of Hedge Funds has to be allocated into the other assets due to the maximum bound. For portfolios with a standard deviation of less than 25% Bonds receive the highest part. Then, for higher levels of standard deviation and mean, Stocks become more attractive to make sure that the higher levels of mean return are achieved. From the x-axis we see that the risk spectrum with the constrained portfolios has been extended. This is primarily caused by REITs and the high weights of Stocks as these two assets exhibit the highest standard deviations.

A more detailed report on the individual differences between the frameworks for each investor type can be found in Appendices B, C, and D.

To sum it up, we can say that for all investors, models and time horizons the maximum admissible fraction of 10% is invested into Hedge Funds. Investors B and C almost always allocate the maximum of 25% in REITs, Investor A stays slightly below. The remainder is allocated to Bonds and Stocks according to the investor’s risk aversion. The fraction of Bonds decreases with the risk aversion and lies between 0% and 49% for the mean-variance, utility- and Score-model, and between 41% and 66% for the \( \Omega \)-model. The reverse is true for stocks, with optimal allocations between 23% and 67% for mean-variance, utility and Score and between 14% and 26% for the \( \Omega \)-model.

Our example investor from Section 6.1 (Investor A, 3 years) now has an \( \Omega \)-optimal portfolio that consists of 10% Hedge Funds, 65% Bonds, 14% Stocks and 11% REITs. All other optimal portfolios put less money into Bonds (44%-49%) and invest more in Stocks (24%-29%) and REITs (17%-22%). This means, that the \( \Omega \)-model puts most of the remainder (after investing 10% in Hedge Funds) in the riskless Bonds, whereas the other models are more diversified.

To compare the results of the different optimization models, we also calculated the probability of outperformance, i.e. the percentage of scenarios in which a model outperformed the other. It turned out, that for the one-year time horizon there is no big difference between the different models (maximum percentage of outperformance of 55%) with the utility model being the best. For the 3 and 5 year time horizon the Score-model clearly outperforms the other models (percentage of outperformance between 63% and 82%).

**Figure 1:** Mean-variance optimal weights (unconstrained vs. constrained) with superimposed efficient frontier of 3 year returns
If we want to compare the optimization models by means of the performance measures Sharpe-ratio, $\Omega$ and Score-value, we can see that the $\Omega$-model performs best in terms of Sharpe-ratio and of course $\Omega$, but worst in terms of Score-value. The highest Score-value is obtained with mean-variance, utility- and Score-model.

7 DYNAMIC OPTIMIZATION WITH MONTHLY REALLOCATIONS

In this section we pick investor A to analyze a different setting: we allow the investor to redistribute the entire portfolio at the end of each month. The setup we used in Section 6 was a static one: even if the market environment develops unfavourably, no reallocations will take place at all. Now, we divide the time from the present until the end of the investment horizon in distinct periods, each period having the length of one month. Thus, we get 12 periods for the 1 year, 36 periods for the 3 years and 60 periods for the 5 years horizon. At the end of each of these periods the investor is allowed to make a new allocation decision.

The approach with reallocation was employed by Grauer and Hakansson (1986), Grauer and Hakansson (1987), Grauer and Hakansson (1995), as well as Hagelin and Pramborg (2005) who employed the power-utility model as described above. The procedure applied by them uses a rolling window of input data while moving through time, i.e. along the return path. We use the same approach here. However, while the mentioned studies use historical data only, we amplify the results by performing the optimization on all the simulated paths from above. The length of the rolling window is set to 24 representing a 2 years time horizon (Hagelin and Pramborg (2005) also report the results of their optimization with a window length of 24, i.e. 2 years. Their results with a 3 years window are similar to the ones with the 2 years window.).

Since we want to compare the results of this dynamic optimization with those obtained above through the static optimization we use the identical investment horizons of 1, 3 and 5 years for the same investor type A. To simplify the analysis we do not consider any transaction costs. Thus, the investor can freely reallocate at the end of each month.

A given series of portfolio returns should be evaluated consistently as good or bad, independent of the model or type of optimization it stems from. Therefore, investor A applies the same values for the risk aversion parameters (of course, the thresholds $\tau$ and the risk free rate are rescaled for the new period length of one month). This setting ensures the comparability between the approaches of static and dynamic optimization.

7.1 Results of dynamic optimization

For each of the three investment horizons we compute the cumulative portfolio returns and calculate the same performance measures as in the previous Section 6 (see Table 10).

We make the following observations for the performances: since we compute $\Omega$ for the cumulative returns at the end of the considered investment horizon the $\Omega$-values do not necessarily have to be the highest for the $\Omega$-framework even though this framework maximizes the $\Omega$-value on the monthly basis. Of course, this is also true for the Score-value. On the considered data the utility-optimized portfolio creates the best $\Omega$-performance for 5 years (427,342). This is mainly due to a very low downside (0.00000364).

When interpreting the results of the static optimization we found high similarities between the allocations and performances of the mean-variance model and the utility model. For the portfolios with monthly reallocations this picture changes slightly. The differences become obvious, especially when performance is measured by $\Omega$ and Score-value although the means and standard deviations (and therefore the Sharpe Ratio) are very similar.

Table 10 summarizes all performance measures and compares them to those of the static optimization. In general, the results confirm what we expected: the performance of the portfolios with monthly reallocations exceeds the performance of the statically optimized ones. There are just a few cases where the performance measures indicate a poorer performance of the dynamically optimized portfolios. To shed light on these observations we produce scatter plots of the annualized returns of the statically and dynamically optimized portfolios (see Figure 2)
In these plots we find the reasons for the observations made above: for the mean-variance, the utility and the Ω-model we find effectively only returns above the line indicating that the result of the dynamically optimized portfolio is almost never below that of the statically optimized portfolio. But for the vast majority of paths the attained return exceeds that of the statically optimized counterpart causing higher levels of mean returns for the dynamically optimized portfolios. In addition, we find that the clouds of dots spread over a larger range for the dynamically optimized portfolios in all case. So the dynamically optimized portfolios above showed a higher standard deviation. Neither effect dominates the other in all cases which causes the Sharpe Ratio to show better performance of the dynamically optimized portfolios not in all cases (when performance is measured by Ω and Score-value, i.e. by incorporating higher moments, all portfolios perform better in the dynamically-optimized case).

We find that all three models produce rather similar plots for all three time horizon: the percentage of scenarios which the statically optimized portfolios succeed in do not increase for longer time horizons. In fact the fraction decreases slightly (see Table 11 below).

If we look at the Score-optimal portfolio, we see that for the returns of 1 year time horizon it behaves like the other models, but when the investment horizon is extended we find that the statically optimized portfolio performs better than the dynamically optimized portfolio for a growing number of scenarios (more than 50% for the 5 year time horizon). We also find that the range of portfolio returns tends to be larger for the statically optimized portfolio (which is contrary to the other three models).

| Model: Mean-var; investment horizon 3 years | Model: Utility; investment horizon 3 years | Model: Omega; investment horizon 3 years | Model: Score; investment horizon 3 years |

Table 11: Percentage of scenarios in which the statically optimized portfolio earns a higher return than the dynamically optimized portfolio

<table>
<thead>
<tr>
<th>Mean-var. model</th>
<th>Utility model</th>
<th>Omega model</th>
<th>Score model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 yr 5.96 % 5.01 % 1.18 % 0.78 %</td>
<td>3 yrs 5.82 % 3.87 % 1.09 % 29.29 %</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 yrs 5.11 % 2.82 % 0.74 % 52.37 %</td>
<td>1 yr 5.96 % 5.01 % 1.18 % 0.78 %</td>
<td></td>
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</tr>
<tr>
<td>3 yrs 5.82 % 3.87 % 1.09 % 29.29 %</td>
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<td>5 yrs 5.11 % 2.82 % 0.74 % 52.37 %</td>
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<td></td>
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<tr>
<td>3 yrs 5.82 % 3.87 % 1.09 % 29.29 %</td>
<td>5 yrs 5.11 % 2.82 % 0.74 % 52.37 %</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7.2 Optimal allocations

Let us now have a look at the weights of the dynamically optimized portfolios. For this purpose we compute the mean of the optimal weight of each asset for each point in time across all 10,000 scenarios (see Figure 3). For all models it is save to say that some time to adjust is needed in which the average of the optimal weights changes heavily. For the longer horizons of 3 and 5 years, respectively, we observe that the average portfolio weights swing into a relatively stable allocation after approx. 2 years (i.e. 24 periods). For reasons of comparison we add the optimal allocation of the static optimizations for each case (see Figure 3).

Since the statically optimized portfolio weights of the mean-variance, utility, and Score models clearly emphasized Hedge Funds and invested heavily into this asset and the dynamically optimized weights on average allocate in all assets, we find the fractions of Hedge Funds to be a lot smaller in the dynamic case (e.g., Investor A invests between 20% after 1 year and 60% after 5 years). The fractions of REITs are found to be a lot higher since on average all dynamically optimized portfolios contain REITs and the statically optimized allocations hardly assign REITs (e.g., Investor A invests between 10% (Score-model) after 1 year and 35% after 3 years). Just the Ω-optimal portfolios led to more balanced allocations in the static optimization. Therefore, the comparison of these weights with the average weights of the dynamically optimized portfolios exhibits that they almost match, especially for the horizons of 3 and 5 years.
Taking all the aspects into consideration, it can be said that REITs and Hedge Funds play a major role in the optimal portfolios. On average all frameworks allocate substantially into both asset classes. The \( \Omega \)-optimal and the Score-optimal portfolios are more dominated by Hedge Funds with fractions of 60% for the horizons of 3 and 5 years and 50% and more for the 1 year horizon while the mean-variance-optimal and the utility-optimal portfolios assign 20-25% to REITs and 45-50% to Hedge Funds (depending on the investment horizon). Due to the more balanced allocations the dynamically optimized portfolios create lower downsides than their statically optimized counterparts. This is intuitive since with monthly reallocations the portfolio is able to face bad developments in single assets with lower allocations.

Again, we take a closer look at the 3 year time horizon for Investor A and his optimal portfolios: for the mean-variance and the utility-model the optimal portfolios are quite similar (Bonds=10%, Stocks=25%, HFs=45%, REITs=20%), whereas the \( \Omega \)- and Score-optimal investors put more money into Hedge Funds (60%) and Bonds (20%) and less in Stocks (10%) and REITs (10%). If we compare this to the results obtained above in the static case, we can see that besides the \( \Omega \)-model, where there are no big changes, in the dynamic case the investor allocates less in Hedge Funds and more into Bonds, Stocks and REITs.

All the results are in line with results from previous studies (see, e.g., Grauer and Hakansson (1995) or Hagelin and Pramborg (2005)) that both, Hedge Funds and REITs are allocated in the optimal portfolio and that these allocations can be remarkably high. Hagelin and Pramborg (2005) found that a risk avers investor heavily invests into hedge funds (for a portfolio of bonds, stocks and hedge funds). They also showed that for a less risk averse investor even undiversified portfolios appear to be optimal which allocate all capital into Hedge Funds for substantial periods of times.

As in the static case we also calculated the probability of outperformance for the different models in the dynamic case. Here, the mean-variance model clearly outperformed all other models and the Score-model is outperformed by all other models.

If we compare the optimization models by means of the performance measures we can not decide which model is best. According to the Sharpe-ratio the \( \Omega \)-model performs best, the highest \( \Omega \)-values are attained by the utility and Score-model and the highest Score-value is reached by mean-variance and utility model.

\section{Dynamic Optimization Under Market Conditions}

The results from Section 7 showed the optimal allocation for an investor in a dynamic setting for different optimization models. In reality such an investor would have to face further constraints and peculiarities that would possibly affect his decisions and hence the allocations shown above. Three of such peculiarities are discussed in this section, namely data biases in hedge funds data, lock-up periods and transaction costs.

\subsection{Data biases}

Due to the peculiarities of the hedge fund markets a couple of biases can be found in hedge funds data. The most common and probably best-analyzed biases are selection, survivorship, and backfill bias.

The first one (selection bias) is caused by the voluntary reports of returns. With no obligation to report bad results, poorly performing funds may decide not to report the returns to data collectors. This results in databases which contain just the well-performing funds. In addition, funds that do not want to grow anymore i.e. are not looking for new investors often do not report their returns to databases any longer. This may cause an opposite effect. Altogether, the selection bias describes the problem that return databases may not adequately represent the true hedge funds universe.

The second bias, survivorship bias, occurs because many data vendors just report returns of funds that are still in operation. Funds which are closed down during that period often are excluded. A major reason for ceasing operations of a fund is bad performance (roughly 30% of newly established hedge funds do not survive the first three years (see p.28 of Brooks and Kat (2002)). Therefore data available from databases tend to underestimate the possible returns (see Kat (2003)). Liang (2000) found that this overestimation exceeds 2% p.a. while Kat (2003) reports 2%-4% p.a.

Backfill biases are caused by filling in historical returns of newly added hedge funds. When a hedge fund is added to an index or included in a database it can choose the entry date. Of course, the fund management chooses a date which makes the performance appear very well. Thus, the backfill bias is an upward bias which is estimated by Fung and Hsieh (2000) to be approx. 1.4 percentage points p.a.

Taking these biases into account we correct the Hedge Funds mean accordingly by subtracting 4% p.a. from the mean return and rerun all the steps described above, i.e. correction of mean level with Black-Litterman, model fit and parameter estimation, computation of state-dependent correlations, simulation, and eventually portfolio optimization. For brevity, we do not report all the steps but focus on the changes in the optimal portfolios: Figure 4 displays the allocations similarly to Figure 3 of the unadjusted optimization for the set which corrects the Hedge Fund mean by 4% p.a.

We find the allocations to Hedge Funds decreased significantly and that of Stocks increased. But similar to above the allocation finds a stable balance after 2 years - of course, with less Hedge Funds in the optimal portfolios. For the \( \Omega \)-model this stable portfolio almost matches the optimal weights of the static optimization.

E.g., for the 3 year time horizon of Investor A we find the following optimal portfolios: for the mean-variance it consists of 10% Bonds, 30% Stocks, 30% HFs and 30% REITs, for the utility-model of 10% Bonds, 30% Stocks,
35% HFs and 25% REITs, for the Ω-model of 25% Bonds, 15% Stocks, 45% HFs and 15% REITs and for the Score-model of 30% Bonds, 10% Stocks, 50% HFs and 10% REITs.

All in all we can say that the sensitivity of the portfolio allocation towards corrections for biases is rather small. Although there are some changes in the optimal portfolios, the structure is quite similar.

8.2 Lock up periods

So far we have not considered that many hedge fund managers establish so called lock up periods. These lock up periods describe a time interval the investor is required to hold the once bought shares at minimum. Since the managers are more or less free in choosing these intervals, we use an average of 1 year.

The model is adjusted in a way that the fraction of the portfolio invested into hedge funds is not included in the monthly optimization within the lock up period. Just after one year the weights of hedge funds will be adjusted to represent the developments of the assets. All other assets are reallocated as before on a monthly basis.

We found that the allocations do not change fundamentally. Again after two years the weights swing into a stable allocation which is almost exactly the same as in the case without lock up periods. Therefore we conclude that the lock up periods do not influence the allocation process to a great extent.

8.3 Transaction costs

Since we excluded transaction costs from our analysis so far an investor could not earn the reported returns in reality. When transaction costs, i.e. costs for buying and selling assets, are considered the returns of the dynamically optimized portfolios will be reduced. Assuming a realistic set of transaction costs (Stocks: 5bps, Bonds: 5bps, REITs: 30bps and HFs: 50bps) led to the result that, in a simple setting of transaction costs proportional to the amount reallocated, the returns of the dynamically optimized portfolios are only reduced by a small amount and are still clearly above the static ones. Furthermore, we found that the average transaction costs differ between the optimization models with the lowest transaction cost for the mean-variance model. E.g., the average discount on returns per month, i.e. the value by which the monthly returns decrease when considering transaction cost, for Investor A on a 3 year time horizon is 0.0037=0.37% for the mean-variance model, 0.0041 for the utility-model, 0.0072 for the Ω-model and 0.0073 for the Score-model.

9 CONCLUDING REMARKS

This study has examined the attractiveness of hedge funds and real estate investment trusts (REITs) with Asian background in a mixed-asset portfolio of Asian bonds and Asian stocks. A lot of prior studies have shown that returns of alternative assets, especially hedge funds, tend to exhibit non-normal behaviour (see, e.g., Brooks and Kat (2002), Amin and Kat (2003), and McFall Lamm (2003)) and that these departures from normality are decisive to the investor.
Analyzing the portfolio weights across all scenarios will be rather low. Hedge Funds (depending on the investment horizon). Optimal portfolios assign 20-25% to REITs and 45-50% to bonds. For investment horizons of 3 and 5 years and 7% REITs for Investor A over 3 years), which leads to the highest Sharpe-ratios and Ω-values.

Comparing the different optimization models by means of outperformance probabilities led to the result that in the static case the utility model (1 year time horizon) and the Score model (3 and 5 year time horizon) are dominant, whereas the mean-variance model appears to be the model of first choice in the dynamic case. We also found that the optimal allocations in the dynamic models swing into a relatively stable allocation after approx. 2 years. It was also shown that these optimal allocations in the dynamic setting are relatively stable towards biases that typically appear in Hedge Funds data.

In this article we found in a static one-period world all three representative investor types to heavily allocate into hedge funds across all portfolio optimization models. For lower degrees of risk aversion not seldom the entire portfolio is invested into hedge funds. Even the models including higher moments (the utility, Ω, and Score frameworks) prefer hedge funds. REITs do not play a major role in the optimal portfolios. The Ω-framework is the only model which allocates REITs in all optimal portfolios. The contradiction to the results of other studies basically grounds on the fact that in our study on Asian alternative investments the sample data of hedge funds exhibits very atypical features which appear to be very attractive to the investor: the skewness is almost 0 and the excess kurtosis atypically low and therefore appealing.

The article also encompasses some kind of dynamic portfolio optimization. Here, we allowed for monthly reallocations of the entire portfolio. We found that for the mean-variance, the utility, and the Ω-model the portfolio return at the end of the investment horizon virtually always exceeded that of the static optimization without monthly reallocations when transaction costs were excluded. For these three models the probability that the statically optimized portfolio earns a higher return as the dynamically optimized portfolio decreased when we increased the investment horizon. The Score-model produced results contrary to those of the other models. For investment horizons of 3 and 5 years we observed considerable numbers of scenarios in which the dynamically optimized portfolio led to a return worse than that of the static counterpart. Furthermore, the percentage of these scenarios increased with increasing horizon length. For a 5 years horizon we obtained more scenarios in which the statically optimized portfolio performed better than scenarios this portfolio performed worse in. One reason might be found in the highest average allocations to Bonds. Since Bonds promise the lowest level of mean return amongst all asset classes considered in the study the respective portfolio return will be rather low.

Analyzing the portfolio weights across all scenarios revealed that Hedge Funds and REITs play a major role in all optimal portfolios. All in all we can say, that on average all frameworks allocate substantially into both asset classes. The Ω-optimal and the Score-optimal portfolios are more dominated by Hedge Funds with fractions of 60% for the horizons of 3 and 5 years and 50% and more for the 1 year horizon while the mean-variance-optimal and the utility-optimal portfolios assign 20-25% to REITs and 45-50% to Hedge Funds (depending on the investment horizon).

We want to add one last issue which we think is of interest for future research: for both optimizations, with and without reallocations, we stated the assumption of no consumption at all during the investment time. In reality, investors do need money at certain times. A popular suggestion to cure this problem is the use of one portfolio for each of the consumption times. Then each portfolio is optimized independently of the others. This may not be optimal from an overall perspective of all portfolios. This issue is tackled in the field of asset-liability-management which incorporates not just the assets in the portfolio consumption process but also the liabilities.

REFERENCES


(see, e.g., Scott and Horvath (1980)). By using univariate Markov switching models we account for many specifics of alternative investment returns or financial asset returns in general: autocorrelation, smoothed data, non-normality, data biases etc. Earlier studies on the aftermaths of the higher moment characteristics of alternative investments, as e.g., Brunner and Hafner (2006), showed that incorporating these features leads the high portfolio fractions of hedge funds in the mean-variance framework decrease due to unfavourably negative skewness and high excess kurtosis.
A PROOF OF COVARIANCE FORMULA

Suppose we want to compute the state-dependent residuals correlations of asset J and asset K. As defined above the return processes of both assets are given by

\[ y_t^J = \mu_{S_t}^J + \Phi^J \cdot (y_{t-1}^J - \mu_{S_{t-1}}^J) + \sigma_{S_t}^J \epsilon_t^J \]

and

\[ y_t^K = \mu_{S_t}^K + \Phi^K \cdot (y_{t-1}^K - \mu_{S_{t-1}}^K) + \sigma_{S_t}^K \epsilon_t^K \]

with \( J \neq K \).

Applying the derivations in Timmermann (2000) the covariance of both processes can be expressed as follows:

\[ \text{Cov}(y_t^J, y_t^K) = \hat{\sigma}^T \cdot \mathbf{E} \left[ (y_t^J - \mu^J) (y_t^K - \mu^K) \right] \]

\[ = \hat{\sigma}^T \cdot \text{E} \left[ (y_t^J - \mu_J^J + \mu_J^K) (y_t^K - \mu_J^K + \mu_J^J) \right] \]

\[ = \hat{\sigma}^T \cdot \left\{ \text{E} \left[ (y_t^J - \mu_J^J) (y_t^K - \mu_J^K) \right] + \mu_J^J \mu_J^K \right\} \]

\[ = \hat{\sigma}^T \cdot \left\{ \text{E} \left[ \sum_{i} (\Phi_i^J \cdot \epsilon_i^J) \cdot \epsilon_i^K \right] \right\} \]

\[ \Rightarrow \text{Cov}(y_t^J, y_t^K) = \hat{\sigma}^T \cdot \text{const} \]

with

- State combinations: \( \tilde{S}_t = (S_t^J, S_t^K) \)

\[ \begin{pmatrix} \rho_{\mu_J^J, \mu_J^K}^J \\ \rho_{\mu_J^J, \mu_J^K}^K \\ \rho_{\mu_J^J, \mu_K^K}^J \\ \rho_{\mu_J^J, \mu_K^K}^K \\ \rho_{\mu_K^J, \mu_K^K}^J \\ \rho_{\mu_K^J, \mu_K^K}^K \end{pmatrix} \]

and

\[ \tilde{S}_t = (S_t^J, S_t^K) \]

where \( S_t^J \) and \( S_t^K \) denote state 1 of asset J and asset K, and \( S_t^J \) and \( S_t^K \) denote state 2 of asset J and asset K, respectively;
• Transition probability matrix:

\[
\widetilde{P} = \begin{bmatrix}
\tilde{p}_{S_1,S_1} & \tilde{p}_{S_1,S_2} & \tilde{p}_{S_1,S_3} & \tilde{p}_{S_1,S_4} \\
\tilde{p}_{S_2,S_1} & \tilde{p}_{S_2,S_2} & \tilde{p}_{S_2,S_3} & \tilde{p}_{S_2,S_4} \\
\tilde{p}_{S_3,S_1} & \tilde{p}_{S_3,S_2} & \tilde{p}_{S_3,S_3} & \tilde{p}_{S_3,S_4} \\
\tilde{p}_{S_4,S_1} & \tilde{p}_{S_4,S_2} & \tilde{p}_{S_4,S_3} & \tilde{p}_{S_4,S_4}
\end{bmatrix}
\]

with e.g. 

\[
\tilde{p}_{S_i,S_i} = \tilde{p}_{j,t} \cdot \tilde{p}_{j,t} = \tilde{p}_{1,1}(t) \cdot \tilde{p}_{1,1}(t)
\]

\[
= p_{11} \cdot p_{11} = (1 - p_{12}) \cdot (1 - p_{12})
\]

• Unconditional state probabilities

\[
\tilde{\pi} = \begin{bmatrix}
\pi_1^* \\
\pi_2^* \\
\pi_3^* \\
\pi_4^*
\end{bmatrix} = \begin{bmatrix}
\pi_1^j \cdot \pi_1^K \\
\pi_2^j \cdot \pi_2^K \\
\pi_3^j \cdot \pi_3^K \\
\pi_4^j \cdot \pi_4^K
\end{bmatrix}
\]

where \(\pi_i^*\) denotes the unconditional probability for the process \(*\) being in state \(i, * \in \{J, K\} \) and \(i \in \{1, 2\}\).

• (A): Substituting backwards into

\[
y_i = \mu_i + \Phi_i \cdot (y_{i-1} - \mu_{i-1}) + \sigma_i \cdot e_i
\]

leads directly to

\[
y_i - \mu_i = \sum_{i=0}^{\infty} \Phi_i \cdot \sigma_i \cdot e_{i-1}
\]

• (B): Assumptions

- \(e_i \sim \text{i.i.d. } N(0,1)\)
- \(\text{Cov}(\epsilon_{i,t}, \epsilon_{j,t}) = 0, \ i \neq j\)
- \(\text{Cov}(\epsilon_{i,t}, \epsilon_{j,t}^c) = \rho_{ij}^c, \ \forall i, \forall t\)

• (C): Definitions of \(\mu_i^j, \mu_i^K, \sigma_i^j, \sigma_i^K\) and \(\star_i^j, \star_i^K\)

\[
\mu_i^j = \mu_i^j, \mu_i^K, \sigma_i^j, \sigma_i^K
\]

and

\[
\star_i^j = \star_i^j, \star_i^K
\]

with \(* \in \{J, K\}\).

• (D): Property of state-steady probabilities

\[
\tilde{\pi}^T \tilde{P}^j = \tilde{\pi}^T
\]

• Notations:

- \(\otimes\) element-by-element multiplication operator,
- \(\mu_i^j\) state-independent overall mean of asset \(J\),
- \(\rho_{ij}^c\) state-dependent residuals correlations of asset \(J\) and asset \(K\).

### B Static Results for Investor A

As we stated above, the \(\Omega\)-model shows profoundly different optimal allocations than the other frameworks due to the fact that Investor A puts about 25% of the portfolio value into Bonds for all investment horizons if he or she invests according to the \(\Omega\) model. The other models prefer Hedge Funds and Stocks, especially for longer periods. Their allocations are more alike.

We have a more thorough look at the differences between the performance of the \(\Omega\) framework and the other frameworks. Figure 5 shows the scatter-plot of the return series of the utility-optimal and \(\Omega\)-optimal portfolios for 3 years. For each of the 10,000 paths one point is plotted. The black line marks the locations for which the return of both portfolios would be the same. Scenarios in which the utility framework earned a higher return as the \(\Omega\) framework lie below that line and vice versa. First of all, we find that the cloud of dots stretches out along the black line. Therefore, either both models come off rather well or rather badly in all scenarios. In the plot we can see that the \(\Omega\)-portfolio leads to better results for low levels of return and that good scenarios tend to be more favourable for the utility-portfolio. The cloud appears to be very dense with just a few scenarios with extreme returns. While at the first glance both return series have almost equal means for each of the time horizons, the variance seems to differ profoundly. The range of (annualized) returns of the utility-portfolio is \([-0.07; 0.33]\) while the returns of the \(\Omega\)-portfolio lie in the interval \([-0.04; 0.25]\).

![Figure 5: Scatter-plot of the returns of the utility- and \(\Omega\)-optimal portfolios](image-url)
Even though the utility-portfolio exhibits a higher mean than the respective Ω-portfolio, its Sharpe Ratio is worse. This is due to the higher standard deviation which diminishes the mean advantage.

We can conclude that the Ω-model looks quite promising for investors of type A. Due to the rather low loss thresholds of this investor type (4.06% to 4.42% p.a.) the model tries to use Bonds to generate less risky returns and invests in all assets rather than concentrating all money in one single asset. As the Ω-measure is computed by dividing the upside through the downside potential this appears to be a good strategy. If the investor was interested in higher returns it would be better to follow another strategy, preferably the Score model. We found that the Ω-model is beaten by all other models in the best scenarios. This comes at the price of an unattractive performance of the other models in worse scenarios.

**C STATIC RESULTS FOR INVESTOR B**

Similar to investor A, investor B’s mean-variance model and the utility model lead to (almost) the same optimal portfolios for all investment horizons. The Score model tends to allocate more aggressively and the Ω model invests again in all assets available. All models allocate into REITs just within the 1 year time horizon. The reason is the low correlation to Stocks and Hedge Funds and therefore a low standard deviation of the whole portfolio. In all unconstrained portfolios the biggest part is assigned to Hedge Funds which provide for the highest mean return: The upside created by Hedge Funds is big enough not to care too much about the downside.

But with the application of the constraints, the portfolios cannot create enough upside: The maximum constraint limits the investment into Hedge Funds. The Omega- and Score-models try to limit the downside in order to maximize the portfolio performance. Subsequently, Bonds and REITs are allocated. They have the lowest individual standard deviations, low correlations to Stocks and Hedge Funds, and are even negatively correlated with each other. They also seem promising for measures incorporating higher moments: Both assets show positive skewness and negative excess kurtosis for 3 and 5 years. Therefore, the portfolio downsides can be limited by including them into the optimal allocation.

Recall that Ω is computed by dividing upside through downside potential. Both potentials are determined by applying the individual loss threshold τ. Investor B applies higher thresholds than investor A for all time horizons: 5.47% p.a. for 1 year, 5.42% p.a. for 3 years and 5.37% p.a. for 5 years. Therefore, investor B needs to allocate higher fractions of the portfolio to assets that exceed this return. In addition, we need to consider the downside. This is equally important as the upside within the computation of Ω. We expect portfolios exhibiting reasonably high mean returns above the threshold with, at the same time, having the smallest downside to perform best within the Ω-framework. For this reason, we compare the optimal Ω-portfolio with the more aggressive allocation of the Score model (both for 3 years investment period). We analyze in detail the Ω and Score-value of the portfolios and the respective upsides and downsides:

Since the optimal Score-portfolio is entirely made up of Hedge Funds no diversification effects take place and the standard deviation exceeds that of the Ω-optimal portfolio. As expected, the upside is very high since Hedge Funds promise the highest mean return of 12.4% p.a. As above, we find that the Ω-portfolio performs better for lower levels of returns while the Score-portfolio clearly outperforms the Ω-portfolio in good scenarios. The values of the upsides and downsides confirm this: For both measures, Ω and Score-value, the respective upside of the Ω-portfolio (0.23 and 0.30) is below that of the Score-portfolio (0.27 and 0.34). The Ω-optimal minimizes the downside better than the Score-optimal portfolio. This leads to a better performance according to Ω where the computation puts equal weights to upside and downside.

Within the Score framework the risk aversion parameter determines to what extent the upside is reduced by the downside. The less risk averse an investor is the lower will be the λΩ-value. That softens the importance of the downside and consequently favours strategies with higher upsides. Investor B’s λΩ-value for 3 years is apparently low enough (3.85) to let the Ω-optimal portfolio appear worse-performing in spite of a lower Score-downside. The downside is just not as decisive as it is when performance is measured by Ω.

| Table 12: Statistics of 3 years returns of unconstrained Ω- and Score-portfolios |
|---------------------------------------------|-----------------|-----------------|
| Investor B, 3 years investment horizon     | Score-model     | Omega-model |
| Mean                                        | 0.4280          | 0.3908         |
| Mean (p.a.)                                 | 0.1261          | 0.1162         |
| Standard deviation                          | 0.2014          | 0.1622         |
| Standard dev. (p.a.)                        | 0.2014          | 0.1622         |
| Skewness                                    | 0.4016          | 0.3532         |
| Excess kurtosis                             | 0.2628          | 0.1946         |
| Sharpe ratio                                | 1.6687          | 1.8426         |
| Omega                                       | 0.2710          | 0.2317         |
| Upside                                      | 0.0860          | 0.0039         |
| Omega                                       | 45.5127         | 59.9822        |
| Downside                                    | 0.3381          | 0.2999         |
| Omega                                       | 0.0020          | 0.0010         |
| Upside                                      | 0.3303          | 0.2962         |
We now turn to the least risk averse investor. This investor applies the lowest (absolute) values for the risk aversion parameters. With what has been said above we expect the downside potentials of the portfolios to be of less interest and the main goal to be the maximization of the upsides.

In the unconstrained optimization, all $\Omega$-optimal portfolios still allocate all assets even if Bonds make up just little fractions and Hedge Funds clearly dominate the portfolio. The other frameworks have the most extreme allocations amongst all investors. Again, Stocks are included in the 1 year portfolios due to the high mean within that period. But for 3 and 5 years the portfolios are almost completely invested into Hedge Funds, the asset with the highest mean return.

When constraints are applied the portfolio allocations to Hedge Funds and REITs are set to the maximum limit. The remainder is put into Stocks completely. Just the $\Omega$ model still invests into all four assets. While investor B retains a little fraction of Bonds in the other frameworks for reasons of diversification, investor C looks more for high returns and the maximization of the upside. Thus, Stocks appear to be the most promising asset and with 25% of the portfolio invested into REITs there are altogether three assets for diversification. Bonds are not required for further diversification since the risk aversion is considerably low.

We mentioned above that a less risk averse investor tends to put more effort into the maximization of the upside than into the minimization of the downside when weights are subject to constraints. The more risk averse an investor is the higher will be the impact of the downside to reduce the performance. This explains why the fraction of Bonds (as the asset with the worst mean) is lower in the optimal portfolio of investor C. Also within the Score model investor C emphasizes the maximization of the upside ($\Omega$-upside as well as Score-upside are higher for investor C’s portfolio) even if the downside rises considerably. When the Score-value is computed for investor B we find that the performance of investor C’s more extreme portfolio is worse than his own more balanced portfolio. This is because the downside is weighed higher due to the higher value of $\lambda_{sc}=6.91$. The picture changes completely when the risk aversion of investor C is applied: The lower value of $\lambda_{sc}$ (5.46) reduces the impact of the downside. Therefore the portfolio with the higher upside is preferred and ranked higher.