Asset Allocation with Credit Instruments

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Abstract

This article presents a consistent, scenario-based asset allocation framework for analyzing traditional financial instruments and credit instruments in a portfolio context. Our framework accounts for the distinct return characteristics of credit instruments by incorporating potential defaults into the total return calculation. We generate correlated default times with a Normal Inverse Gaussian one-factor copula. To determine optimal portfolios we use a mean-variance and a conditional value at risk optimization. Applying our framework to the US market, we find that the mean-variance optimization overestimates the benefits of low-rated credit instruments. Though, optimal portfolios always contain a considerable proportion of credit instruments.

Keywords: Capital-Market Scenarios; Normal-Inverse Gaussian One-Factor Copula; Correlated Default Times; CDS; CDS Index; Corporate Bonds; Portfolio Optimization; Conditional Value At Risk Optimization; Asset Allocation.

Introduction

The credit market has experienced an enormous growth over the last years. As part of the credit market, the credit derivatives market is the worldwide fastest growing derivatives market. It expanded from USD 180 billion of outstanding contracts on a total notional amounts basis in 1997 to USD 5,021 billion in 2004 and to estimated USD 8,206 billion in 2006.[1] After a standardization of credit derivative contracts and the introduction of CDS[2] indices, a revolution in terms of liquidity has taken place which is only one reason why credit instruments are very attractive to investors. In addition, credit instruments such as corporate bonds, credit derivatives and securitizations often have an appealing risk-return profile allowing to enhance portfolio return. Furthermore, due to the correlation structure of their returns to those of traditional asset classes such as stocks and government bonds, they offer high potential for diversification. Finally, they allow to manage credit risk exposure. Even knowing the potential benefits of different credit instruments, investors still have to know how to combine them optimally with traditional asset classes, i.e. they have to decide on the optimal proportion of credit related products, especially for their individual level of risk-aversion.

A reasonable asset allocation including credit instruments needs to account for the distinct return characteristics of these instruments. We exemplarily analyze the return properties of daily log-returns of US Lehman aggregates over the period from 7^{th} of November, 2002 to 29^{th} of September, 2006. The descriptive statistics of the returns are summarized in Table 1 where it becomes obvious that the risk-return profile of credit instruments cannot be sufficiently described by mean and variance alone. We observe exclusively negatively skewed return distributions and positive excess kurtosis indicating non-normal distribution of return data. This is confirmed by Jarque-Bera tests for normality. The non-normality might be attributed to defaults within the aggregates. The logical consequence is that the shape of the return

[1] See [British Bankers' Association (2004)] and [FitchRatings (2005)].

[2] A credit default swap (CDS) is a contract in which the protection buyer pays a fixed periodic fee on the notional amount to the protection seller over a predetermined period. In exchange, the protection buyer receives a contingent payment from the protection seller, triggered by a credit event of a reference entity.

distributions should be considered in a realistic asset allocation framework.

In this article, we propose a consistent, scenario-based asset allocation framework that is composed of a simulation, a total return calculation, and an optimization framework. With the simulation framework we simulate consistent capital-market scenarios for interest rates, credit spreads and returns of equity indices, on the one hand, and correlated default times, on the other hand. For the scenario generation, we use the four-factor model according to [Schmid, Zagst, Antes and Ilg (2006b)] and the model suggested by [Zagst, Meyer and Hagedorn (2006)]. To generate non-normal return distributions of credit instruments, we simulate correlated default times for issuers with different ratings. In this context, the one-factor copula approach for modeling correlated default times between reference entities has become very popular. We use this approach for a Normal-Inverse-Gaussian (NIG) one-factor copula as suggested by [Kalemanova, Schmid and Werner (2007)] since it is able to produce more realistic properties for default times than the wide-spread Gaussian version, e.g. a higher probability of joint defaults of different companies.

After having determined the return distribution of the government bonds, the equity index and the credit instruments, we are able to calculate optimal asset allocations according to different optimization criteria. We do not only apply traditional mean-variance portfolio optimization according to [Markowitz (1952)] but also conditional value at risk (CVaR) optimization. The latter is able to better capture the distinct distributional properties of credit instruments as it takes into account the left tail of a return distribution. We can show that investors benefit from adding credit related products to their portfolio, i.e. with the same level of risk they can generate a higher expected return.

The article is organized as follows. The second section describes the simulation framework for the risk factors and the correlated default times. In the third section, we give the general conditions for the total return calculations. In particular, we explain the pricing of funded CDS and funded CDS indices. The applied optimization framework is presented in the fourth section. In the fifth section, we provide the model parameters and present the simulation and optimization results. We close in the last section with a summary of the main results.

Simulation Framework

The simulation framework consists of a model to simulate risk factors on the one hand and correlated default times on the other hand.

First, we introduce a model to simulate risk factors. We simulate the short rate, the growth rate of the Gross Domestic Product (GDP), and the short-rate spread with the extended model of Schmid and Zagst (see [Schmid, Zagst, Antes and Ilg (2006b)]). The short inflation and the returns of an equity index are simulated with the integrated market model suggested by [Zagst, Meyer and Hagedorn (2006)]. The latter can be embedded into the extended model of Schmid and Zagst.

In the following, we assume that markets are frictionless and perfectly competitive, that trading takes place continuously, that there are no taxes, transaction costs, or informational asymmetries, and that investors act as price takers. We fix a terminal time horizon T^*. Uncertainty in the financial market is modeled by a complete probability space $(\Omega, \mathcal{G}, \mathbf{Q})$ and all random variables and stochastic processes introduced below are defined on this probability space. We assume that $(\Omega, \mathcal{G}, \mathbf{Q})$ is equipped with three filtrations $\mathbb{H}$, $\mathbb{F}$, and $\mathbb{G}$, i.e. three increasing families of sub-σ-fields of $\mathcal{G}$. The default time τ of an obligor is an arbitrary non-negative random variable on $(\Omega, \mathcal{G}, \mathbf{Q})$. For the sake of convenience we assume that $\mathbf{Q}(\tau = 0) = 0$ and $\mathbf{Q}(\tau > t) > 0$ for every $t \in (0, T^*]$. For a given default time τ, we introduce the associated default indicator or hazard function $H(t) = \mathbf{1}_{\{\tau \leq t\}}, t \in (0, T^*]$. Let $\mathbb{H} = (\mathcal{H}_t)_{0 \leq t \leq T^*}$ be the filtration generated by the process H. In addition, we define the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$ as the filtration generated by the multi-dimensional standard Brownian motion $W'(t) = (W_r(t), W_w(t), W_u(t), W_s(t))$ and $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T^*}$ as the enlarged filtration $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, i.e. for every t we set $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$. All filtrations are assumed to satisfy the usual conditions of completeness and right-continuity. For the sake of simplicity we furthermore assume that $\mathcal{F}_0$ is trivial. It should be emphasized that τ is not necessarily a stopping time with respect to the filtration $\mathbb{F}$ but of course with respect to the filtration $\mathbb{G}$. If we assumed that τ was a stopping time with respect to $\mathbb{F}$, then it would be necessarily a predictable stopping time.

We assume that there exists a measure $\hat{\mathbf{Q}} \sim \mathbf{Q}$ such that all discounted price processes of the financial instruments are martingales relative to $(\hat{\mathbf{Q}}, \mathbb{G})$. We will assume throughout that for any $t \in (0, T^*]$ the $\sigma-$fields $\mathcal{F}_{T^*}$ and $\mathcal{H}_t$ are conditionally independent (under $\hat{\mathbf{Q}}$) given $\mathcal{F}_t$. Following [Bielecki and Rutkowski (2004), p. 167 and p. 242], this is equivalent to the assumption that for any $t \in (0, T^*]$ and any $\hat{\mathbf{Q}}-$integrable $\mathcal{F}_{T^*}-$measurable random variable X we have $E^{\hat{\mathbf{Q}}}[X|\ \mathcal{G}_t] = E^{\hat{\mathbf{Q}}}[X|\ \mathcal{F}_t]$. The modeling takes already place after measure transformation, i.e. under $\hat{\mathbf{Q}}$ with the money-market account $B(t) = e^{\int_0^t r(l)dl}$ as numéraire and $r(t)$ denoting the non-defaultable short rate. In the following, all processes are defined on the probability space $(\Omega, \mathcal{G}, \hat{\mathbf{Q}})$.

The dynamics of the non-defaultable nominal short rate r is described by a two-factor Hull-White model and is given by

$$dr(t) = (\theta_r(t) + b_r\omega(t) - a_r r(t))\,dt + \sigma_r dW_r(t), \tag{1}$$

$$d\omega(t) = (\theta_\omega - a_\omega\omega(t))\,dt + \sigma_\omega dW_\omega(t), \tag{2}$$

where $a_r, b_r, \sigma_r, a_\omega, \sigma_\omega$ are positive constants, θ_ω is a non-negative constant, $\theta_r(t)$ is a continuous, deterministic function[3] and $dW_r(t)$ and $dW_\omega(t)$ are independent standard Brownian motions. ω represents the GDP growth rate so that the interest-rate levels directly depend on general economic conditions. The dynamics of the short-rate spread are described by

$$ds(t) = [\theta_s + b_{su}u(t) - b_{s\omega}\omega(t) - a_s s(t)]\,dt + \sigma_s dW_s(t), \tag{3}$$

$$du(t) = [\theta_u - a_u u(t)]\,dt + \sigma_u dW_u(t), \tag{4}$$

where $b_{su}, b_{s\omega}, a_s, \sigma_s, a_u, \sigma_u$ are positive constants, θ_s, θ_u are non-negative constants and $dW_u(t), dW_s(t)$ are independent standard Brownian motions. u is the so-called uncertainty index and can be interpreted as an aggregation of all available information about the quality of the firm, i.e. it represents the firm-specific risk. The

[3]By a Nelson-Siegel regression, this function ensures that the model fits the market term structure at simulation start date. For details see, for example, [Hull and White (1994)] and [Nelson and Siegel (1987)].

higher its value, the lower the firm's quality. The GDP behaves reversely. If it grows at a higher rate, spreads usually tighten as the probability of default of a firm becomes smaller. We need the dynamics of the short inflation i for the modeling of the equity-return. It is given by

$$di(t) = (\theta_i - a_i i(t))\, dt + \sigma_i dW_i(t), \tag{5}$$

where a_i, σ_i are positive constants, θ_i is a non-negative constant and $dW_i(t)$ is a standard Brownian motion. Then, the continuous return of an equity index $R_E(t)$ is described by

$$dR_E(t) = [\alpha_E + b_{E\omega}\omega(t) - b_{Ei}i(t) + b_{ER}r(t)]\, dt + \sigma_E dW_E(t), \tag{6}$$

where $\alpha_E \in \mathbb{R}, b_{E\omega}, b_{Ei}, b_{ER}, \sigma_E$ are positive constants and $dW_E(t)$ is a standard Brownian motion. Note, that this process reflects the so-called "stock return-inflation puzzle" stating that inflation negatively influences stock returns.[4]

So far, we saw the dynamics of the SDEs (1) – (4) under the real measure Q. Though, as we are interested in zero-coupon bond prices as well as interest rates and credit spreads for different terms, we need to know the parameters of the SDEs under the risk-neutral equivalent martingale measure $\hat{Q}$. Replacing a_r, a_ω, a_s, a_u by $\hat{a}_r, \hat{a}_\omega, \hat{a}_s, \hat{a}_u$ and using independent standard Brownian motions $d\hat{W}_r(t), d\hat{W}_\omega(t), d\hat{W}_s(t), d\hat{W}_u(t)$ instead of $dW_r(t), dW_\omega(t), dW_s(t), dW_u(t)$, leads to the processes under this measure. The relationship between the parameters is given by $\hat{a}_k = a_k + \lambda_k \sigma_k^2$, where $k = r, \omega, s, u$. We obtain λ_r, λ_ω, λ_s, λ_u when changing measure from Q to $\hat{Q}$ by applying Girsanov's Theorem.[5]

The processes introduced earlier imply analytical formulas for the zero rates at time t to maturity T of non-defaultable and defaultable bonds, $R(t,T)$ and $R^d(t,T)$, as well as for the credit spreads, $S(t,T)$. So we can determine the complete term structure. The explicit formulas can be found in the Appendix.

[4]See for example [Arin and Mamun (2004)], [Sellin (2001)], [Feldstein (1980)], [Fama (1981)] and [Friedman (1977)].

[5]See, for example, [Schmid, Zagst, Antes and Ilg (2006b)].

To simulate correlated default times we use Li's approach, who presents an efficient algorithm in [Li (2000)]. We use it with a one-factor copula as applied for example by [Hull and White (2004)]. The one-factor copula is a simple, but powerful way to quickly define a correlation structure between several variables. Therefore, it has become very popular and the standard approach in practice to simulate correlated default times. The underlying idea is the following: In reality, more firms default during a recession than during a booming period. This implies that each firm is subject to the same set of macroeconomic environment and that there exists a dependence among the firms. The Gaussian one-factor copula is often applied due to an easy implementation and the appealing properties of a standard normal distribution, such as the stability under convolution. We rather use the NIG copula which is able to overcome some modeling deficiencies of the Gaussian copula, e.g. the lack of tail dependence. The NIG distribution is a mixture of normal and inverse Gaussian distributions. It is a four parameter distribution with very interesting properties. It can produce fat tails and skewness, it is stable under convolution (under certain conditions), and the density function, the distribution function and the inverse distribution function can be computed sufficiently fast.[6] A definition of the NIG distribution and its most important properties can be found in [Kalemanova, Schmid and Werner (2007)].

Before presenting the copula model, we explain the idea of the Large Homogeneous Portfolio (LHP) approach, introduced by [Vasicek (1987)]. It assumes a constant default correlation structure over the reference credit portfolio, with the same default probabilities and the same recovery rate in case of default, and it models default using a one-factor Gaussian copula. [Kalemanova, Schmid and Werner (2007)] modified the LHP model by replacing the Gaussian distribution with the NIG distribution. Their one-factor copula model is briefly introduced in the following.

Consider a homogeneous portfolio of m credit instruments. The standardized asset return of the i^{th} issuer in the portfolio, A_i, is assumed to be of the form

$$A_i = aM + \sqrt{1-a^2}X_i \tag{7}$$

[6] See [Kalemanova, Schmid and Werner (2007)] and [Kalemanova and Werner (2006)].

with independent random variables

$$M \quad \sim \quad \mathcal{NIG}\left(\alpha,\, \beta,\, -\frac{\beta\gamma^2}{\alpha^2},\, \frac{\gamma^3}{\alpha^2}\right), \tag{8}$$

and

$$X_i \quad \sim \quad \mathcal{NIG}\left(\frac{\sqrt{1-a^2}}{a}\alpha,\, \frac{\sqrt{1-a^2}}{a}\beta,\, -\frac{\sqrt{1-a^2}}{a}\frac{\beta\gamma^2}{\alpha^2},\, \frac{\sqrt{1-a^2}}{a}\frac{\gamma^3}{\alpha^2}\right), \tag{9}$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$.

Then, the asset returns A_i also follow a NIG distribution with the parameters

$$A_i \sim \mathcal{NIG}\left(\frac{\alpha}{a},\, \frac{\beta}{a},\, -\frac{1}{a}\frac{\beta\gamma^2}{\alpha^2},\, \frac{1}{a}\frac{\gamma^3}{\alpha^2}\right) =: \mathcal{NIG}_A \tag{10}$$

and A_i has zero mean and unit variance. The factor M represents the systematic common market factor and X_i represents firm-specific factors. Equation (7) defines a correlation structure between the random variables A_i. Then, the correlation between the asset returns of two issuers is given by a^2, in the case of a homogeneous portfolio. Conditional on M the asset returns of different issuers are independent.

Let us assume that default at time t occurs when the asset return of obligor i falls below the threshold $C(t)$, i.e. $A_i \leq C(t)$. Using this copula model, the variable A_i is then mapped to default time τ_i of the i^{th} issuer with a percentile-to-percentile transformation as described for example by [Hull and White (2004)] or [Bluhm (2003)]:

$$Q(t) = \mathbb{P}[\tau_i \leq t] = \mathbb{P}[A_i \leq C(t)] = \mathcal{NIG}_A(C(t)). \tag{11}$$

$Q(t) = \mathbb{P}[\tau_i \leq t]$ is the real-world distribution of default times estimated with a migration matrix following [Bluhm (2003)]. According to (11) we conclude that $C(t) = \mathcal{NIG}_A^{-1}(Q(t))$ and thus

$$Q(t) = \mathbb{P}[\tau_i \leq t] = \mathbb{P}[A_i \leq \mathcal{NIG}_A^{-1}(Q(t))] = \mathbb{P}[Q^{-1}(\mathcal{NIG}_A(A_i)) \leq t].$$

Now, we can simulate A_i according to (7) and determine the default times via $\tau_i = Q^{-1}(\mathcal{NIG}(A_i))$.

Framework for Total Return Calculation

Based on the simulations according to the previous section, we price the following financial instruments: Government and corporate coupon bonds, funded CDS and funded CDS indices. The pricing for all instruments is done under the following conditions. We price the instruments at every simulated timestep $t_k, k \in \{0, ..., p\}$, with $t_0 < t_1 < ... < t_p = T^{sim}$, where T^{sim} denotes the end date of the simulation. We assume that the cash flows occurring during the lifetime of the instrument are reinvested in the respective instrument. In the case of a default, we assume a constant recovery rate REC of 40% and an immediate recovery payment. The cash flows at default time τ are invested in a risk-free cash account and they are compounded at every simulated timestep with the default risk-free government rate. We always assume an initial investment of one unit.

The pricing and total return calculation of the bonds and the equity index is straightforward. The pricing of the funded CDS and the funded CDS index, however, is not trivial and is therefore derived in the following.

A CDS can be issued in a funded version. Then, the protection seller (investor) buys a floating rate note (FRN) which pays a coupon of the 3 month LIBOR plus the fixed CDS spread on a quarterly basis. If no credit event occurs, the coupon is paid until maturity. In this case, the investor receives the notional amount of the FRN. If, however, a credit event of the reference entity takes place, the protection seller receives the recovery value and the contract terminates. The functionality of a funded CDS index is similar to funded CDS. It can be considered as a portfolio of funded CDS. If a credit event of a reference entity takes place, the protection seller receives the recovery value, but the notional amount of the FRN is reduced by the weight of the reference entity (for example from 100% to 99.2% if the reference portfolio contains 125 names[7]) and future coupon payments are based on the new notional amount. Though, the coupon rate remains unchanged.

We consider CDS and the CDS index from a protection seller's perspective. Hence, we view these instruments as investment rather than as a means of hedging. As

[7] The broadest and most actively traded investment-grade indices are the *CDX.NA.IG* for North America and the *iTraxx Europe* for Europe. Both contain 125 equally weighted names.

there are investors who are for regulatory restrictions or due to internal investment policies not allowed to enter into unfunded credit-derivative contracts, we use the funded version of CDS and the CDS index.

Before explaining the total return components of a these instruments, we briefly describe the relationship between the relevant timesteps. At every simulated timestep, $t_k \in \{0, ..., T^{sim}\}$, we price the funded CDS/CDS index. Furthermore, we assume that $t_1^c < ... < t_n^c = T$ denote the spread payment dates, where t_1^c denotes the next spread payment date following the current pricing day, t_k, and T denotes the maturity of the CDS/CDS index. We denote the previous spread payment date or the settlement date, if the first spread payment has not yet been made, by t_0^c. Spread payments are made in arrear – at time t_i^c for the payment period from t_{i-1}^c to t_i^c.

A funded CDS/CDS index has the following total return components: Present value PV_{CDS} of the CDS/CDS index for every pricing day t_k, resulting from possible changes in the default intensity λ_k at t_k[8] and from changes in the interest-rate curve, present value PV_{FRN} of the pure, default risk-free FRN at t_k paying LIBOR, coupon payments PV_S on the notional amount at t_0^c, comprising LIBOR and the fixed spread S^{CDS}, compounded to the following simulated date t_k, and recovery payments PV_{REC} at default time τ, compounded to the following simulated date t_k.

To determine the present value of the funded CDS/CDS index at t_k, we need to know the expected loss at t_k up to every spread payment day, t_i^c, $i \in \{1, ..., n\}$, the premium leg and the protection leg. With the constant default intensity model, we calculate the expected loss at t_k up to t_i^c according to $EL(t_k, t_i^c, \lambda_k) = 1 - \exp(-\lambda_k(t_i^c - t_k))$. This model assumes a constant default intensity, λ_k, for a given time t_k.[9] The premium leg is the present value of all expected spread payments. It is calculated according to

$$\text{Prem Leg}(t_k, T, S^{CDS}, \lambda_k) = \sum_{i=1}^{n} \Delta t_i^c \, S^{CDS} \left(1 - EL\left(t_k, t_i^c, \lambda_k\right)\right) P(t_k, t_i^c),$$

[8] At inception of a funded CDS/CDS index, the fixed spread is determined so that PV_{CDS} is equal to zero. During the term of the contract, PV_{CDS} can be positive or negative.

[9] This implies that the partial recovery of market value in the extended model of Schmid and Zagst is stochastic over time.

where $\Delta t_i^c = t_i^c - t_{i-1}^c$, S^{CDS} is the fixed annual spread of the CDS/CDS index, $P(t_k, t_i^c)$ is the discount factor and $1 - EL(t_k, t_i^c, \lambda_k)$ denotes the probability of no default up to time t_i^c.

The protection leg is the present value of all expected protection payments made by the protection seller. As we assume a constant recovery rate REC we can calculate the protection leg by

$$\begin{aligned}\text{Prot Leg}(t_k, T, \lambda_k) &= (1 - REC) \int_{t_k}^{T} P(t_k, l)\, dEL(t_k, l, \lambda_k) \\ &\approx (1 - REC) \sum_{i=1}^{n} \left(EL(t_k, t_i^c, \lambda_k) - EL(t_k, t_{i-1}^c, \lambda_k) \right) P(t_k, t_i^c).\end{aligned}$$

At issuance of the funded CDS, the fixed annual spread S^{CDS} is determined so that the value of the premium leg equals the value of the protection leg:

$$S^{CDS} = \frac{(1 - REC) \sum_{i=1}^{n} \left(EL(t_k, t_i^c, \lambda_k) - EL(t_k, t_{i-1}^c, \lambda_k) \right) P(t_k, t_i^c)}{\sum_{i=1}^{n} \Delta t_i^c \left(1 - EL\left(t_k, t_i^c, \lambda_k\right)\right) P(t_k, t_i^c)}. \tag{12}$$

The equality of premium leg and protection leg as shown in equation (12) must also hold for every day, when S^{CDS} is substituted by the quoted par yield spread.[10] After this substitution, we can solve the equation for λ_k and determine the constant default intensity at t_k.

Let $N(t_k, \tau)$ denote the notional amount of a funded CDS/CDS index at timestep t_k. Then, the present value of the CDS/CDS index is given by

$$PV_{CDS}(t_k, T, S^{CDS}, \lambda_k) = \left(\text{Prem Leg}(t_k, T, S^{CDS}, \lambda_k) - \text{Prot Leg}(t_k, T, \lambda_k)\right) N(t_k, \tau). \tag{13}$$

The calculation of the present values of the other total return components is straightforward. Then, we determine the total return of the investment, R^{CDS}, for every

[10]For the sake of simplicity we use the simulated zero spread as the quoted par yield spread. The absolute difference between the spreads, however, is very low. We analyzed historical credit spreads from 31^{st} of December, 1991 to 29^{th} of September, 2006. The average difference in this period was at most 1.5 bps for rating classes AA, A2 and BBB and maturities of up to 5 years. The worst case on a single day was 4.9 bps.

$t_k > 0$ by

$$R^{CDS}(t_{k-1}, t_k) = \frac{PV_{CDS}(t_k, T, S^{CDS}, \lambda_k) + PV_{FRN}(t_k) + PV_{REC}(t_k) + PV_S(t_k)}{PV_{CDS}(t_{k-1}, T, S^{CDS}, \lambda_{k-1}) + PV_{FRN}(t_{k-1}) + PV_{REC}(t_{k-1})} - 1.$$

Optimization Framework

For all optimization criteria, we make the following assumptions. There are neither transaction costs nor taxes; all securities can be divided arbitrarily; the portfolios remain unchanged over time; there are n given assets to invest in with returns R_i, $i = 1, ..., n$; the expected return of asset i is given by $\mu_i := \mathbb{E}[R_i]$ and $\mu := (\mu_1, ..., \mu_n)^T$; x_i is the portfolio weight of asset i with $\sum_{i=1}^n x_i = 1$, and the portfolio is denoted by $x := (x_1, ..., x_n)^T$.

Mean-Variance Optimization

The mean-variance optimization can be viewed as traditional portfolio optimization and is based on the model of [Markowitz (1952)]. The basic assumption is that investors select their portfolios taking into account only the first two moments of the asset returns – mean and variance – and the correlation between the assets.
Let the covariance matrix be denoted by $C = (c_{ij})_{i,j=1,...,n}$, where $c_{ij} = Cov[R_i, R_j]$, and let the variance be denoted by $\sigma_i^2 := c_{ii} > 0$. Then, the mean-variance optimization is given by the following optimization problem

$$\begin{aligned} \min_x \quad & x^T C x \\ s.t. \quad & \mu^T x \geq \overline{\mu},\ \mathbf{1}^T x = 1,\ x \geq 0, \end{aligned} \tag{14}$$

where $\mathbf{1} = (1, \ldots, 1)^T$. If we solve the optimization problem (14) for every possible $\overline{\mu}$, we obtain the set of all efficient portfolios.
Obviously, the optimization problem (14) only takes into account mean and variance of a portfolio. This is appropriate, for example, if returns are normally distributed. However, the returns of credit instruments are not normally distributed, as seen earlier. Therefore, this criterium is only of limited use for the optimization of portfolios

including credit instruments.

CVaR Optimization

The conditional value at risk (CVaR) represents the expected value of all losses that exceed a certain value at risk (VaR). Formally, we can define the CVaR as follows. Let $1-\alpha$ be the confidence level for the VaR with $\alpha \in (0,1)$. Then, the CVaR of a portfolio's return $R(x)$ is given by

$$\begin{aligned} \text{CVaR}(x,\alpha) &= -\mathbb{E}[R(x)|R(x) \leq C_R(\alpha)] \\ &= -\mathbb{E}[R(x)|R(x) \leq -\text{VaR}(x,\alpha)], \end{aligned} \tag{15}$$

where $C_R(\alpha)$ denotes the α-quantile of the portfolio's return distribution $R(x)$. From the formulas above, it becomes evident that the CVaR provides information on the negative tail of a return distribution since it is not only focussed on the α-quantile but also takes into account the shape of its tail. This is of great importance if instruments may default and so produce fat tails. The corresponding CVaR optimization is given by

$$\begin{aligned} \min_x \quad & \text{CVaR}(x,\alpha) \qquad (16) \\ s.t. \quad & \mu^T x \geq \overline{\mu},\ \mathbf{1}^T x = 1,\ x \geq 0. \end{aligned}$$

Model Calibration and Simulation Results

We fit our model to market data as of 30^{th} of September, 2006 (simulation start date). To calibrate the simulation model, we use parameters estimated by [Schmid, Zagst and Antes (2006a)] and [Zagst (2006)]. The parameters for the short rate and the GDP growth rates are given by $a_r = 0.37867$, $\hat{a}_r = 0.24782$, $b_r = 0.13315$, $\sigma_r = 0.01496$, $a_\omega = 1.18532$, $\hat{a}_\omega = 0.26847$, $\theta_\omega = 0.01583$, $\sigma_\omega = 0.00601$. The credit spread parameters are displayed in Table 2. We adjust the estimated parameters $\hat{a}_s$, $\hat{a}_u$ and b_{su} to better meet historical data in terms of average spreads and spread ranges. For the inflation process and the process for the equity-index returns we use the following parameters: $a_i = 0.64073$, $\hat{a}_i = 0.50319$, $\theta_i = 1.04790$, $\sigma_i = 0.01447$. α_E = -3.07791, b_{ER} = 3.94000, b_{Ei} = 9.32436, $b_{E\omega}$ = 5.10643, σ_E = 0.16000. For the equity index, we adjust the level of the returns and the standard deviation to better reflect the updated historical data.

We want the US CDS index to represent the current composition of the *CDX.NA.IG* in terms of proportion of different rating classes. However, we assume the US CDS index (short CDX) only to be composed of the rating classes *AA*, *A*, and *BBB* with the weights 12.28%, 39.47%, and 48.25%.[11]

For the simulation of correlated default times we use the average one-year migration matrix for the United States, from 1981 to 2005, provided by [Standard & Poor's (2006)]. The parameters for the one-factor copula as of simulation start date are estimated in accordance with the model introduced by [Kalemanova, Schmid and Werner (2007)]. We use the following parameters $a = 0.37029$, $\alpha = 0.70138$, $\beta = 0$. So, the correlation a of a single reference entity to the common market factor is equivalent to a correlation between two reference entities of $a^2 = 0.13711$.

Having simulated 5000 scenarios on a quarterly basis with government interest rates, credit spreads and correlated default times, we can now price various financial instruments. All bonds, CDS and the CDX are assumed to have an initial term of approximately 5 years.

Table 3 reports the return characteristics for investment horizons 1 year and 3 years where we make some general observations. We see higher expected returns and higher standard deviations or levels of CVaR with decreasing credit quality of the bonds. Besides, we observe that – as soon as defaults have occurred – the distributions of credit instruments become strongly non-normal.[12] For the 1 year investment horizon the AA and A-rated CDS and the CDX have considerably less volatility and a lower CVaR than government and corporate bonds. At the same time, CDS and the CDX have a similar expected return to government bonds. Though, comparing the 1 year and the 3 year investment horizon, we make an interesting observation.

[11] See `www.fitchratings.com`. We added the *AAA* rating class to rating class *AA* and *BB* to *BBB*, as there are only parameters for the rating classes *AA*, *A*, and *BBB* available. Furthermore, we normalized the weights to sum up to one.

[12] This can be discovered with the statistics skewness, excess kurtosis and minimum.

For the 3 years horizon, the CDS have a similar or higher risk than the corporate bonds and a lower return. This phenomenon can be easily explained by the nature of the instruments. Fixed-income instruments are exposed to market risk which is the higher, the longer the time to maturity. CDS and the CDX are very similar to a FRN and only have a very small market risk. They pay the short-term interest rate which is lower than the medium-term interest rates if the curve is normal. In addition, the outcome of interest rates is more volatile over a longer investment horizon.

In Table 4, we show the linear correlations between the returns of the financial instruments for the investment horizons 1 year and 3 years. We make the following main observations: Government-bond and corporate-bond returns have a high positive correlation. This is reasonable as the main driver of the bond return is the government rate which is the same for all bonds. Differences in bond returns come form different credit spreads and different default times. Bond and equity-index returns are negatively correlated. Such a correlation can also be empirically observed. There is a small positive or negative correlation between returns of corporate bonds and funded CDS/CDX. This may seem counterintuitive at first sight. A closer look at the nature of these instruments proves otherwise. We need to differentiate between two opposing effects which can be attributed to single return components of the funded CDS/CDX. On the one hand, the present value of the CDS PV_{CDS} (see equation (13)) strongly behaves like the price of the corporate bond. Interest rates being equal, PV_{CDS} and the price of the corporate bond decrease if spreads increase resulting in a high positive correlation. On the other hand, bonds are fixed income instruments with prices that strongly depend on the level of interest rates. If interest rates are falling, the bond returns increase as bond prices increase. For the FRN linked to the CDS/CDX it is the other way round. Since its return strongly depends on the floating LIBOR, the return of the FRN tends to decrease with falling interest rates resulting in a high negative correlation between bond returns and returns of funded CDS/CDX. All in all, the two opposing effects cancel each other out to a certain degree resulting in a small positive or negative correlation between corporate bonds and funded CDS/CDX. The returns of the equity index and CDS or the CDX are positively correlated. This is reasonable since both equity-index returns and the 3 month government rate are to a large proportion driven by the short rate, which was empirically confirmed by [Zagst, Meyer and Hagedorn (2006)].

To conclude, we identify appealing risk-return profiles and low correlations between bonds, the equity index and CDS/CDX. Therefore, investors should benefit from holding a portfolio consisting of traditional financial instruments and credit instruments.

After having analyzed the return characteristics, we can turn to portfolio optimization. For this purpose, we consider a 1 year and a 3 year investment horizon and the following three investment universes: Initial investment universe consisting of government bonds and an equity index; extension by corporate bonds; extension by CDS and the CDX.

Mean-Variance Approach

With a mean-variance optimization according to the optimization problem (14) for the relevant investment horizons and investment universes we obtain the efficient frontiers and the resulting asset allocations displayed in Figure 1. There, we can identify some general structures: At first, we examine the efficient frontiers. Allowing for corporate bonds in the portfolio optimization leads to an upward shift of the efficient frontier compared to the initial investment universe. For the same level of risk, a higher expected return can be generated. Allowing additionally for CDS and the CDX leads to a shift to the left of the efficient frontier, i.e. an investor can reduce the portfolio risk. The proportion of the potential risk reduction or return enhancement depends on the investment horizon. For a 1 year horizon the minimum-variance portfolio of the initial investment universe has an expected return of 5.76% and a standard deviation of 3.04%. If we allow for all instruments, the corresponding portfolio with the same expected return of 5.76% has a standard deviation of only 0.70% and the portfolio with the same standard deviation of 3.04% has an expected return of 6.72%. We see similar effects for a 3 year investment horizon.

Now, we focus on the mean-variance optimal asset allocations. In the initial investment universe, optimal allocations only consist of government bonds and an equity index. In the largest investment universe, government bonds are substituted to a large extent with corporate bonds, CDS and the CDX. As already seen from Table 3, AA and A-rated CDS and the CDX have a similar expected return to government bonds but a much lower standard deviation for a 1 year horizon. Therefore, particularly for lower levels of standard deviation, optimal portfolios contain a considerable proportion of CDS and the CDX. For levels of standard deviation higher than 7.64% (1 year horizon) and 10.38% (3 years horizon), an optimal portfolio only consists of corporate bonds and an equity index. For lower levels of standard deviation, corporate bonds are partially replaced by CDS and the CDX. There is only a rather small proportion of government bonds for a low level of risk, due to the risk-return profile of government bonds and CDS.

Conditional Value at Risk

In the CVaR optimization, we assume α to be 1%, i.e. we consider the mean of the worst 1% of the portfolio return as risk measure. The resulting efficient frontiers and the optimal asset allocations are displayed in Figure 2.

When comparing the results of the CVaR optimization with those of the mean-variance optimization, the first impression is very similar. In the following, we describe the main results of the CVaR optimization at first. Then, we describe and explain some differences of mean-variance and CVaR optimization. We begin with the efficient frontiers. Adding credit instruments to a portfolio so far only consisting of the initial investment universe, leads to an upward shift and a shift to the left of the efficient frontier. So, there is an enormous potential for return enhancement or risk reduction. The potential depends on the investment horizon. It is exemplarily shown for the minimum-CVaR portfolio. For a 1 year investment horizon the minimum-CVaR portfolio of the initial investment universe has an expected return of 5.79% and a CVaR of 2.10%. If we allow for all instruments, the corresponding portfolio with the same expected return of 5.79% has a CVaR of only -3.12%. Fixing the CVaR to 2.10%, the expected return of an optimal portfolio in the largest investment universe is 6.78%. The effects are similar for a 3 year investment horizon.

Next, we analyze the optimal asset allocations for an investor using the CVaR-criterium. In an optimal portfolio, government bonds are partially substituted with corporate bonds, CDS and the CDX compared to the initial investment universe. As explained earlier, AA and A-rated CDS and the CDX have a similar expected return to government bonds, but a lower CVaR for a 1 year horizon. Particularly for low levels of CVaR, corporate bonds are partially substituted with CDS and the CDX. For levels of CVaR higher than 18.29% (1 year horizon) and 14.47% (3 years horizon), an optimal portfolio only consists of corporate bonds and an equity index. If, however, lower levels of CVaR are of interest, optimal allocations are composed of a considerable proportion of credit derivatives.

Having a closer look at the resulting optimal allocations of mean-variance and CVaR optimizations, we can identify some differences. For a 1 year investment horizon, optimal allocations contain CDS even for high levels of expected returns. While mean-variance optimal portfolios are composed of CDS up to an expected portfolio return of 7.91%, CVaR optimal portfolios should allocate into CDS up to a return level of 8.94%. In contrast, CVaR optimal portfolios contain significantly less BBB-rated, but significantly more AA-rated corporate bonds. For a 3 year investment horizon, CVaR optimal portfolios hold CDS up to an expected portfolio return of 25.30%, while mean-variance optimal portfolios only consist of CDS up to returns of 21.53%. Furthermore, the former are composed of a significant larger proportion of government bonds than the latter. Again, we observe a considerably lower proportion of BBB-rated corporate bonds for the CVaR optimal portfolios. BBB-rated bonds provide a rather high expected return. Though, the standard deviation is not able to adequately capture the tail events. From these observations we can conclude that the mean-variance optimization underestimates the benefits of credit derivatives, particularly for longer investment horizons, by only considering the first two moments of the return distribution, and not its tail. Moreover, it overestimates the benefits of BBB-rated corporate bonds by only looking at the expected return and volatility but mainly ignoring the tail events. Our analyses show that an investor, optimizing his portfolio with either the mean-variance or the CVaR criterium, can add performance to his portfolio for a given level of risk, or he can reduce risk for a given target return level. This can be realized due to a low correlation between the different instruments and their attractive risk-return profile.

Comparison of Selected Optimal Portfolios

We close our analyses of the simulation results with a comparison of selected optimal portfolios. We examine the optimal asset allocation for two representative investors which we denote by risk-averse and risk-affine. Their benchmark portfolio is defined on the initial investment universe. The former has a benchmark portfolio with 30% equity index and 70% government bonds while the latter has a benchmark portfolio with 70% equity index and 30% government bonds. In the Tables 5 and 6 we display the risk-return profile of the benchmark portfolio and the optimal portfolios. In addition, we show the optimal allocations for the investment universe allowing for all instruments and having the same standard deviation and CVaR, respectively, as the benchmark portfolio. The data for the mean-variance and CVaR-optimal portfolios is taken from the optimization output presented earlier. Here "risk" denotes the respective risk measure, i.e. the standard deviation or CVaR. Beginning with general observations, we see similar resulting expected portfolio returns and portfolio allocations for mean-variance and CVaR optimization. The highest proportion of the equity index can always be observed with the CVaR optimization. The proportion of the equity index for mean-variance optimization is not substantially lower than with CVaR. Still, the proportion of the equity index always resembles the proportion of the benchmark portfolio. Optimal portfolios never allocate in government bonds. For the 1 year horizon, the optimizations lead to a proportion of approximately 11-12% of CDS for the risk-averse investor. A risk-averse investor applying the CVaR criterium and having 3 year investment horizon, should still allocate approximately 12% into CDS. In contrast, mean-variance optimal portfolios do not contain CDS, i.e. mean-variance optimization underestimates the benefit of credit derivatives for longer investment horizons. This effect was already explained earlier.

To sum up, comparing the resulting optimal asset allocations for a representative risk-averse and risk-affine investor, we see that, independent of investment horizon and optimization criterium, an investor always benefits from substituting government bonds by corporate bonds, CDS and the CDX. The resulting optimal allocations, however, strongly depend on the investor type, the optimization criterium and the investment horizon.

Summary and Conclusion

Constructing portfolios with credit instruments requires an appropriate asset allocation framework in order to account for the distinct return characteristics of these instruments, such as non-normality of the return distribution due to potential defaults. We have presented a consistent, scenario-based asset allocation framework which is able to determine optimal portfolios consisting of traditional instruments and credit instruments. The entire framework is composed of a simulation, a total return calculation, and an optimization framework. We have suggested a model to simulate consistent capital-market scenarios offering analytical formulas for the whole term structure of interest rates and credit spreads. Furthermore, we have introduced a model to simulate correlated default times with a NIG one-factor copula which is an extension of the Gaussian one-factor copula. The NIG version is appealing since it allows to generate more realistic properties of default times such as a higher probability of joint defaults of different issuers. After the introduction of general conditions for the total return calculations, we have explained how to price CDS. For portfolio optimization we have applied two criteria – the mean-variance and the CVaR optimization. The former can be viewed as traditional portfolio optimization. Its main disadvantage is that it only takes into account the first two moments of a return distribution which is obviously not appropriate when also allowing for credit instruments. The latter is able to overcome this drawback since it considers the left tail of a distribution. So, it particularly takes defaults into account. Finally, we have presented the model parameters and we have applied our asset allocation framework to the US market for a 1 year and a 3 year investment horizon. We have found that credit instruments have an appealing risk-return profile and a correlation structure providing a considerable potential for diversification. Moreover, we have found that mean-variance optimization overestimates the benefits of low-rated bonds by only focussing on mean and variance, and mainly ignoring the tail events.

Realistic modeling of return characteristics is very important when instruments with specific return characteristics are included in an asset allocation. Otherwise, optimization results are not more than an approximation. Our model provides realistic return distributions. Furthermore, it can be fitted to market data and it can be easily extended by other credit instruments. Therefore, it is appropriate for an asset allocation with credit instruments.

Appendix

The zero rates $R(t,T)$ and $R^d(t,T)$ at time t to maturity T of non-defaultable and defaultable bonds as well as the credit spreads $S(t,T)$ are given by

$$R(t,T) = -\frac{1}{T-t}\left[A(t,T) - B(t,T)r(t) - D(t,T)\omega(t)\right],$$

$$\begin{aligned} R^d(t,T) &= -\frac{1}{T-t}\big[A^d(t,T) - B(t,T)r(t) - D^d(t,T)\omega(t) \\ &\qquad -E^d(t,T)u(t) - F^d(t,T)s(t)\big], \end{aligned}$$

$$\begin{aligned} S(t,T) &= -\frac{1}{T-t}\big[A^d(t,T) - A(t,T) - (D^d(t,T) - D(t,T))\omega(t) \\ &\qquad -E^d(t,T)u(t) - F^d(t,T)s(t)\big], \end{aligned}$$

with

$$A(t,T) = \int_t^T \left[\frac{1}{2}(\sigma_r^2 B(l,T)^2 + \sigma_\omega^2 D(l,T)^2) - \theta_r(l)B(l,T) - \theta_\omega D(l,T)\right] dl,$$

$$\begin{aligned} A^d(t,T) &= \frac{1}{2}\int_t^T \left[\sigma_r^2 B(l,T)^2 + \sigma_\omega^2 D^d(l,T)^2 + \sigma_u^2 E^d(l,T)^2 + \sigma_s^2 F^d(l,T)^2\right] dl \\ &\quad - \int_t^T \left[\theta_r(l)B(l,T) + \theta_\omega D^d(l,T) + \theta_u E^d(l,T) + \theta_s F^d(l,T)\right] dl, \end{aligned}$$

$$B(t,T) = \frac{1}{\hat{a}_r}\left(1 - e^{-\hat{a}_r(T-t)}\right),$$

$$D(t,T) = \frac{b_r}{\hat{a}_r}\left(\frac{1-e^{-\hat{a}_\omega(T-t)}}{\hat{a}_\omega} + \frac{e^{-\hat{a}_\omega(T-t)} - e^{-\hat{a}_r(T-t)}}{\hat{a}_\omega - \hat{a}_r}\right),$$

$$D^d(t,T) = D(t,T) - \frac{b_{s\omega}}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_\omega(T-t)}}{\hat{a}_\omega} + \frac{e^{-\hat{a}_\omega(T-t)} - e^{-\hat{a}_s(T-t)}}{\hat{a}_\omega - \hat{a}_s}\right),$$

$$E^d(t,T) = \frac{b_{su}}{\hat{a}_s}\left(\frac{1-e^{-\hat{a}_u(T-t)}}{\hat{a}_u} + \frac{e^{-\hat{a}_u(T-t)} - e^{-\hat{a}_s(T-t)}}{\hat{a}_u - \hat{a}_s}\right),$$

$$F^d(t,T) = \frac{1}{\hat{a}_s}\left(1 - e^{-\hat{a}_s(T-t)}\right).$$

For an explicit derivation of the zero rates, we refer to [Schmid, Zagst, Antes and Ilg (2006b)].

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US Aggregate	**No. of Issuers as of 29.09.2006**	**Mean (ann.)**	**Volatility (ann.)**	**Skewness**	**Excess Kurtosis**
Aaa	3560	3.50%	3.27%	-0.1310	1.7812
Aa	759	4.23%	4.18%	-0.2944	1.5802
A	1404	5.30%	4.55%	-0.1787	1.4967
Baa	1215	6.99%	4.72%	-0.0508	1.3991
Corporates	2721	5.65%	4.55%	-0.1979	1.2648

Table 1: Descriptive statistics of US Lehman aggregates

	AA	A2	BBB1
a_s	2.96099	2.80727	2.41739
$\hat{a}_s$	2.96099	2.80727	2.41739
θ_s	0.00275	0.00251	0.00238
σ_s	0.00389	0.00309	0.00287
$b_{s\omega}$	0.07373	0.07634	0.09369
a_u	0.11269	0.11057	0.11048
$\hat{a}_u$	0.02180	0.03210	0.04075
θ_u	0.12173	0.18566	0.18980
σ_u	0.00459	0.00476	0.00489
b_{su}	0.98404	0.92214	1.26089

Table 2: Parameter estimation for processes s and u for rating classes AA, $A2$, $BBB1$

1 Year	Gov.	Equity Index	AA	A	BBB	CDS Index	CDS AA	CDS A	CDS BBB
Mean	5.55%	10.43%	6.03%	6.19%	6.37%	5.49%	5.75%	5.92%	6.10%
Median	5.50%	8.25%	6.01%	6.18%	6.51%	5.52%	5.77%	5.95%	6.27%
σ	3.16%	19.30%	3.35%	3.57%	4.73%	0.74%	1.13%	1.70%	3.46%
CVaR(1%)	2.55%	32.14%	3.24%	5.34%	17.71%	-2.60%	-2.64%	-0.39%	13.24%
Min	-4.35%	-43.08%	-59.17%	-58.22%	-58.17%	-13.27%	-57.44%	-57.72%	-57.28%
Max	17.01%	88.10%	17.18%	18.12%	19.63%	7.59%	8.46%	8.54%	10.00%
Skewness	0.0900	0.5568	-1.3820	-3.3200	-6.7629	-5.1385	-34.6157	-30.6476	-16.6226
Excess kurtosis	0.0276	0.4571	28.2614	59.8199	90.1579	109.6775	1927.6047	1135.5933	298.5225
3 Years	Gov.	Equity Index	AA	A	BBB	CDS Index	CDS AA	CDS A	CDS BBB
Mean	16.57%	33.44%	18.04%	18.55%	19.08%	16.43%	17.20%	17.73%	18.41%
Median	16.58%	24.29%	18.16%	18.86%	19.99%	16.46%	17.25%	17.93%	19.10%
σ	2.67%	52.74%	3.86%	5.34%	8.77%	3.33%	4.18%	5.43%	8.06%
CVaR(1%)	-9.41%	54.45%	-1.53%	14.83%	53.12%	-5.77%	0.65%	15.09%	51.90%
Min	7.32%	-67.75%	-55.18%	-55.41%	-55.20%	-30.08%	-54.21%	-54.58%	-54.77%
Max	25.13%	439.46%	27.26%	28.12%	30.60%	27.25%	29.06%	30.19%	32.05%
Skewness	-0.0188	1.3790	-9.2575	-9.8637	-7.1960	-0.9070	-7.3939	-8.4513	-7.1697
Excess kurtosis	-0.0459	3.8584	172.4228	131.3360	56.2670	10.8976	123.0516	107.1262	59.9592

Table 3: Key statistics of total returns of financial instruments

1 Year	Gov.	Equity Index	AA	A	BBB	CDS Index	CDS AA	CDS A	CDS BBB
Gov.	1.0000	-0.1184	0.9508	0.8824	0.6828	-0.2338	-0.1569	-0.1013	-0.0510
Equity Index	-0.1184	1.0000	-0.1182	-0.1110	-0.0835	0.1124	0.0751	0.0639	0.0104
AA	0.9508	-0.1182	1.0000	0.8391	0.6511	-0.2227	-0.0781	-0.0986	-0.0465
A	0.8824	-0.1110	0.8391	1.0000	0.6002	-0.1934	-0.1385	-0.0417	-0.0482
BBB	0.6828	-0.0835	0.6511	0.6002	1.0000	-0.0104	-0.1061	0.0265	0.0560
CDS Index	-0.2338	0.1124	-0.2227	-0.1934	-0.0104	1.0000	0.2652	0.2802	0.1830
CDS AA	-0.1569	0.0751	-0.0781	-0.1385	-0.1061	0.2652	1.0000	0.1107	0.0632
CDS A	-0.1013	0.0639	-0.0986	-0.0417	0.0265	0.2802	0.1107	1.0000	0.0395
CDS BBB	-0.0510	0.0104	-0.0465	-0.0482	0.0560	0.1830	0.0632	0.0395	1.0000
3 Years	Gov.	Equity Index	AA	A	BBB	CDS Index	CDS AA	CDS A	CDS BBB
Gov.	1.0000	-0.1629	0.7001	0.5086	0.3271	-0.4689	-0.3812	-0.2919	-0.1814
Equity Index	-0.1629	1.0000	-0.1068	-0.1014	-0.0593	0.2651	0.2201	0.1764	0.1045
AA	0.7001	-0.1068	1.0000	0.3543	0.2511	-0.2249	-0.2387	-0.1934	-0.0579
A	0.5086	-0.1014	0.3543	1.0000	0.2027	-0.1772	-0.1930	-0.0712	-0.0726
BBB	0.3271	-0.0593	0.2511	0.2027	1.0000	-0.0816	-0.1103	-0.0901	-0.0209
CDS Index	-0.4689	0.2651	-0.2249	-0.1772	-0.0816	1.0000	0.6809	0.5481	0.4063
CDS AA	-0.3812	0.2201	-0.2387	-0.1930	-0.1103	0.6809	1.0000	0.4113	0.2732
CDS A	-0.2919	0.1764	-0.1934	-0.0712	-0.0901	0.5481	0.4113	1.0000	0.2070
CDS BBB	-0.1814	0.1045	-0.0579	-0.0726	-0.0209	0.4063	0.2732	0.2070	1.0000

Table 4: Correlation of total returns of financial instruments

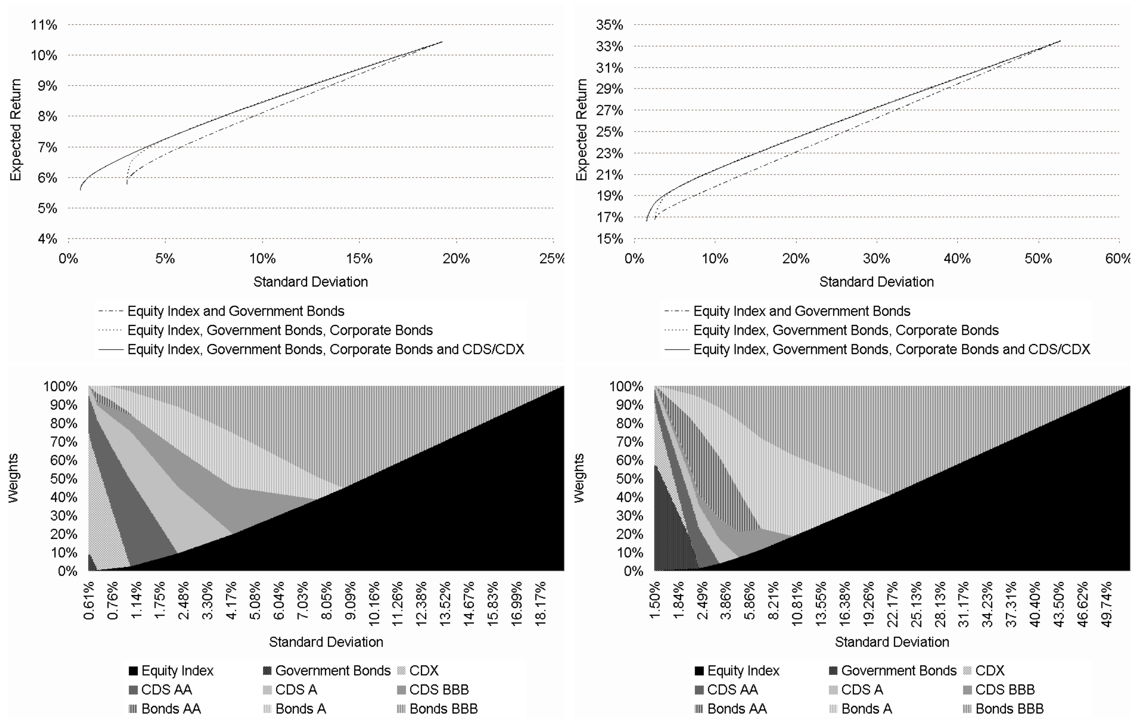

Figure 1: Results of mean-variance optimization, 1 year investment horizon (left-hand side) and 3 year investment horizon (right-hand side)

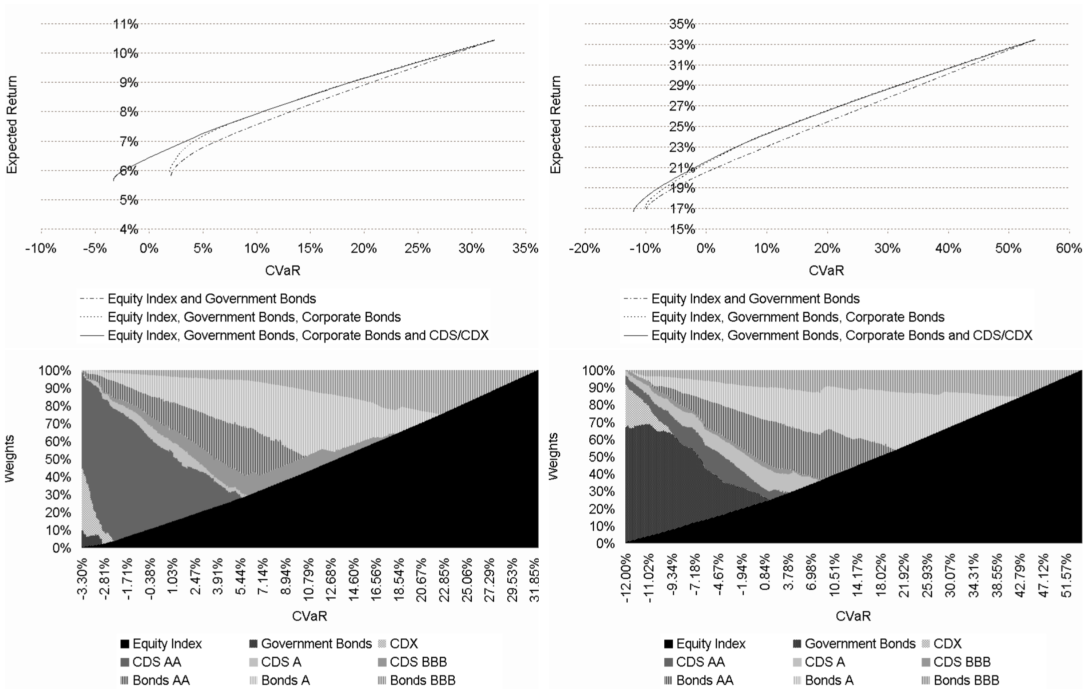

Figure 2: Results of CVaR optimization, 1 year investment horizon (left-hand side) and 3 year investment horizon (right-hand side)

NIG, 1Y	MV av.	CVaR av.	MV aff.	CVaR aff.
risk^B	5.95%	6.52%	13.43%	20.52%
return^B	7.01%	7.01%	8.97%	8.97%
risk^x	5.95%	6.52%	13.43%	20.52%
return^x	7.50%	7.47%	9.21%	9.20%
Gov.	0.00%	0.00%	0.00%	0.00%
Equity Index	29.50%	31.01%	69.82%	69.93%
Corp. AA	0.00%	24.05%	0.00%	0.00%
Corp. A	20.38%	27.10%	0.00%	7.16%
Corp. BBB	37.94%	6.46%	30.18%	22.90%
CDS Index	0.00%	0.00%	0.00%	0.00%
CDS	0.00%	11.39%	0.00%	0.00%

Table 5: Results for risk-averse and risk-affine investor, 1 year investment horizon

NIG, 3Y	MV av.	CVaR av.	MV aff.	CVaR aff.
$risk^B$	15.63%	4.52%	36.80%	32.76%
$return^B$	21.63%	21.63%	28.38%	28.38%
$risk^x$	15.63%	4.52%	36.80%	32.76%
$return^x$	23.13%	22.88%	29.12%	29.18%
Gov.	0.00%	0.00%	0.00%	0.00%
Equity Index	29.04%	30.14%	69.89%	70.89%
Corp. AA	0.00%	24.80%	0.00%	0.00%
Corp. A	23.32%	21.86%	0.00%	15.86%
Corp. BBB	47.64%	11.20%	30.11%	13.25%
CDS Index	0.00%	0.00%	0.00%	0.00%
CDS	0.00%	12.00%	0.00%	0.00%

Table 6: Results for risk-averse and risk-affine investor, 3 year investment horizon